International Journal of Current Research in Science and Technology
Volume 3, Issue 8 (2017), 11–18.
ISSN: 2394-5745
Available Online: http://ijcrst.in/

International Journal of Current Research in Science and Technology

Totally Regular Fuzzy Graphs

Research Article

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Abstract: In this paper, some properties of total degree and totally regular fuzzy graphs are discussed. They are illustrated through various examples. It is proved that every fuzzy graph is an induced subgraph of a totally regular fuzzy graph. The procedure described in the proof is illustrated through an example. Also the total degree of a vertex in fuzzy graphs formed by the operation Union in terms of the total degree of vertices in the given fuzzy graphs for some particular cases are obtained. Using them, their totally regular property is studied.

MSC: 03E72, 05C72.

Keywords: Total degree of a vertex, Complement of fuzzy graph, Complete graph.

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1. Introduction

Fuzzy graph theory was introduced by Azriel Rosenfeld in 1975 [9]. Bhattacharya [1] gave some remarks on fuzzygraphs.Some operations on fuzzy graphs were introduced by Mordeson.J.N. and Peng.C.S. [4]. Zadeh 1965 [3] introduce a mathematical frame work to describe the phenomena of uncertainty in real life situation has been suggested. Research on the theory of fuzzy sets has been witnessing an exponential growth; both within mathematics and in its applications. This ranges from traditional mathematical subjects like logic topology, algebra, analysis etc. to pattern recognition, information theory, artificial intelligence, operations research, neural networks and planning etc. The operations of union, join, Cartesian product and composition on two fuzzy graphs were defined by Mordeson.J.N. and Peng.C.S[4]. The degree of a vertex in some fuzzy graphs and the Regular property of fuzzy graphs which are obtained from two given fuzzy graphs using the operations union, join, Cartesian product and composition was discussed by Nagoorgani. A and Radha. K. [8]. In this paper we study about some properties of totally regular fuzzy graphs. First we go through some basic definitions which can be found in [1-13].

2. Basic Definitions

Definition 2.1 ([2]). *A fuzzy subset of a set V is a mapping σ from V to [0, 1]. A fuzzy graph G is a pair of functions $G : (\sigma, \mu)$ where σ is a fuzzy subset of a non empty set V and μ is a symmetric fuzzy relation on σ, i.e., $\mu(uv) = \sigma(u) \wedge \sigma(v)$. The underlying crisp graph of $G : (\sigma, \mu)$ is denoted by $G^* : (V, E)$, where $E \subseteq V \times V$.*

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Definition 2.2 ([2]). *$G' : (\sigma', \mu')$ is a fuzzy sub graph or a partial fuzzy sub graph of $G : (\sigma, \mu)$ if $\sigma' \subseteq \sigma$ and $\mu' \subseteq \mu$; that is if $\sigma'(u) \leq \sigma(u)$ for every $u \in V$ and $\mu'(uv) \leq \mu(uv)$ for every $uv \in E$.*

Definition 2.3 ([2]). *$G' : (\sigma', \mu')$ is a fuzzy spanning sub graph of $G : (\sigma, \mu)$ if $\sigma' = \sigma$ and $\mu' \subseteq \mu$; that is if $\sigma'(u) = \sigma(u)$ for every $u \in V$ and $\mu'(uv) \leq \mu(uv)$ for every $uv \in E$.*

Definition 2.4 ([2]). *For any fuzzy subset ν of V such that $\nu \subseteq \sigma$, the fuzzy sub graph of $G : (\sigma, \mu)$ induced by ν is the maximal fuzzy sub graph of $G : (\sigma, \mu)$, that has fuzzy vertex set ν and it is the fuzzy sub graph $H : (\nu, \tau)$ where $\tau(u, v) = \tau(u) \wedge \tau(v) \wedge \mu(u, v)$ for all u, v in V.*

Definition 2.5 ([7]). *Let $G : (\sigma, \mu)$ be a fuzzy graph on $G^* : (V, E)$. The degree of a vertex u is $d_G(u) = \sum_{u \neq v} \mu(uv)$. The minimum degree of G is $\delta(G) = \wedge \{d_G(v), \ \forall\ v \in V\}$ and the maximum degree of G is $\Delta(G) = \vee \{d_G(v), \ \forall\ v \in V\}$.*

Definition 2.6 ([5]). *The order and size of a fuzzy graph G are defined by $O(G) = \sum_{u \in V} \sigma(u)$ and $S(G) = \sum_{uv \in E} \mu(uv)$.*

Definition 2.7 ([7]). *Let $G : (\sigma, \mu)$ be a fuzzy graph on $G^* : (V, E)$. Then σ is a constant function if and only if the following are equivalent:*

1. *G is a regular fuzzy graph.*
2. *G is a totally regular fuzzy graph.*

Note: Throughout this paper $G_1^* : (\sigma_1, \mu_1)$ and $G_2^* : (\sigma_2, \mu_2)$ denote two fuzzy graphs with underlying crisp graphs $G_1^* : (V_1, E_1)$ and $G_2^* : (V_2, E_2)$ with $|V_i| = p_i$, $i = 1, 2$. Also $d_{G_i^*}(u_i)$ denotes the degree of u_i in G_i^*.

Definition 2.8 ([8]). *The union of two fuzzy graphs G_1 and G_2 is defined as a fuzzy graph $G = G_1 \cup G_2 : (\sigma_1 \cup \sigma_2, \mu_1 \cup \mu_2)$ on $G^*(V, E)$ where $V = V_1 \cup V_2$ and $E = E_1 \cup E_2$ with*

$$(\sigma_1 \cup \sigma_2)(u) = \begin{cases} \sigma_1(u), & \text{if } u \in V_1 - V_2 \\ \sigma_2(u), & \text{if } u \in V_2 - V_1 \\ \sigma_1(u) \vee \sigma_2(u), & \text{if } u \in V_2 \cap V_1 \end{cases}$$

$$(\mu_1 \cup \mu_2)(e) = \begin{cases} \mu_1(e), & \text{if } e \in E_1 - E_2 \\ \mu_2(e), & \text{if } e \in E_2 - E_1 \\ \mu_1(e) \vee \sigma_2(e), & \text{if } e \in E_2 \cap E_1 \end{cases}$$

Definition 2.9 ([7]). *Let $G : (\sigma, \mu)$ be a fuzzy graph. The degree of a vertex u in G is defined by*

$$d_G(u) = \sum_{u \neq v} \mu(uv) = \sum_{uv \in E} \mu(uv)$$

Definition 2.10 ([8]). *Let $G : (\sigma, \mu)$ be a fuzzy graph on G^*. The total degree of a vertex $u \in V$ is defined by*

$$td_G(u) = \sum_{u \neq v} \mu(uv) + \sigma(u) = d_G(u) + \sigma(u).$$

If each vertex of G has the same total degree k, then G is said to be a totally regular fuzzy graph of total degree k or a k-totally regular fuzzy graph.

Notation 2.11. *The relation $\sigma_1 \leq \mu_2$ means that $\sigma_1(u) \leq \mu_2(e)\ \ \forall\ \ u \in V_1$ and $\forall\ \ e \in E_2$ where σ_1 is a fuzzy subset of V_1 and μ_2 is a fuzzy subset of E_2.*

Lemma 2.12 ([8]). *If $G_1 : (\sigma_1, \mu_1)$ and $G_2 : (\sigma_2, \mu_2)$ are two fuzzy graphs such that $\sigma_1 \leq \mu_2$ then $\sigma_2 \geq \mu_1$.*

Definition 2.13 ([7] Total degree of a vertex). *Let $G : (\sigma, \mu)$ be a fuzzy graph on $G^* : (V, E)$. The total degree of a vertex $u \in V$ is defined by $td_G(u) = \sum_{u \neq v} \mu(uv) + \sigma(u)$.*

Example 2.14. *Consider the following fuzzy graph $G : (\sigma, \mu)$.*

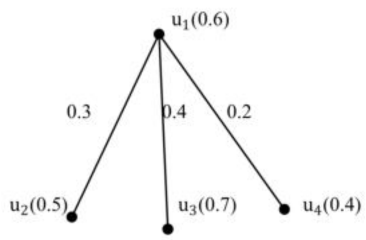

Figure 1.

$$td_G(u_1) = [\mu(u_1u_2) + \mu(u_1u_3) + \mu(u_1u_4)] + s(v) = [0.3 + 0.4 + 0.2 + 0.6] = 1.5$$

Similarly,

$$td_G(u_2) = 0.8; \ \ td_G(u_3) = 1.3; \ \ td_G(u_4) = 0.6$$

3. Totally Regular Fuzzy Graph

Let $G : (\sigma, \mu)$ be a fuzzy graph on $G^* : (V, E)$. If each vertex in G has same total degree k, then G is said to be a totally regular fuzzy graph or k-totally regular fuzzy graph.

Example 3.1. *Consider the following fuzzy graph $G : (\sigma, \mu)$.*

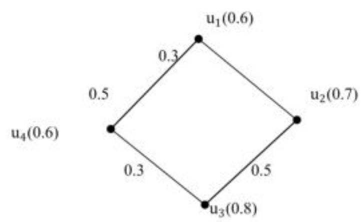

Figure 2.

The fuzzy graph in Figure 2 is a 1.2-totally regular fuzzy graph. Also it is a 0.6-regular fuzzy graph.

Example 3.2.

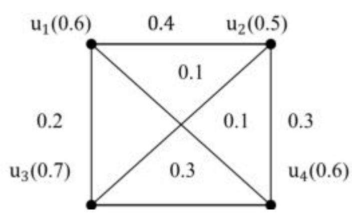

Figure 3.

The fuzzy graph in Figure 3 is a 1.3-totally regular fuzzy graph. But it is not a regular fuzzy graph.

Example 3.3.

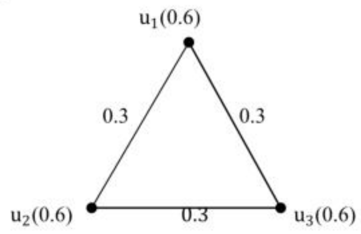

Figure 4.

The fuzzy graph in Figure 4 is a 0.8-regular fuzzy graph. But it is not a totally regular fuzzy graph.

Example 3.4.

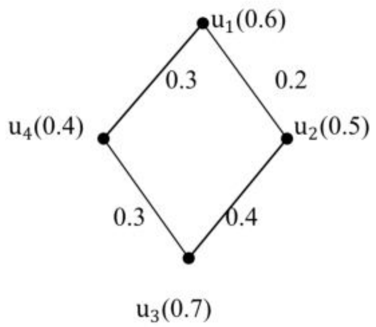

Figure 5.

The fuzzy graph in fig.5 is neither regular nor totally regular fuzzy graph.

Remark 3.5. *From the above examples, it is clear that in general there does not exist any relationship between regular fuzzy graphs and totally regular fuzzy graphs.*

Theorem 3.6 ([7]). *Let $G : (\sigma, \mu)$ be a fuzzy graph on $G^* : (V, E)$. Then σ is a constant function if and only if the following are equivalent :*

(1). G is a regular fuzzy graph.

(2). G is a totally regular fuzzy graph.

4. Properties of Totally Regular Fuzzy Graphs

Theorem 4.1. *In any fuzzy graph G, if $\sigma(v) > 0$ for every vertex $v \in V$, then $td(v) > 0$, for every vertex $v \in V$.*

Proof. Since $\sigma(v) > 0$ for every vertex $v \in V$, $td(v) > 0$, for every vertex $v \in V$. □

Theorem 4.2. *The maximum total degree of any vertex in a fuzzy graph with p vertices is p.*

Proof. For any vertex v,

$$td_G(v) = \sum_{uv \in E} \mu(uv) + s(v) = \sum_{uv \in E} 1 + 1 = d_G^*(v) + 1 = [p-1] + 1 = p.$$

□

Theorem 4.3. *The total degree of a vertex v is $\sigma(v)$ if and only if the degree of v is 0.*

Proof. The total degree of a vertex v is

$$td_G(v) = s(v) \Leftrightarrow \sum_{uv \in E} \mu(uv) + s(v) = s(v) \Leftrightarrow \sum_{uv \in E} \mu(uv) = 0 \Leftrightarrow d_G(v) = 0.$$

□

Corollary 4.4. *The total degree of a vertex v is $\sigma(v)$ for every vertex v in G if and only if G is a null fuzzy graph.*

Proof. $td_G(v) = s(v)$, for every vertex $v \in V \Leftrightarrow d_G(v) = 0$, for every vertex $v \in V \Leftrightarrow G$ is a null fuzzy graph. □

Theorem 4.5. *Every fuzzy graph is an induced fuzzy subgraph of a totally regular fuzzy graph.*

Proof. Let $G : (V, E)$ be any fuzzy graph with p vertices and q edges. If G is totally regular, there is nothing to prove. Suppose that G is not totally regular. Let $\Delta^t = \max\{td(v)/v \in V\}$. Let us prove that G is an induced fuzzy subgraph of a Δ^t-totally regular fuzzy graph. Take a copy G' of G. Take any vertex v with total degree less than Δ^t. Join it to its copy v' in G'. Assign $\min\left\{\sigma(v), \Delta^t - td_G(v)\right\}$ as the membership value of the edge vv'. Do this for all vertices with $td_{G'}(v) < \Delta^t$. Let the resultant fuzzy graph be G_1. For any vertex v with $td(v) < \Delta^t$. If $\mu(vv') = \min\left\{\sigma(v), \Delta^t - td_G(v)\right\} = \Delta^t - td_G(v)$. Then

$$td_{G_1}(v) = td_G(v) + \mu\left(vv'\right) = td_G(v) + \Delta^t - td_G(v) = \Delta^t$$

And all the vertices which have total degree Δ^t in G and their copies in G' will have the same total degree Δ^t in G_1. Also,

$$td_{G_1}\left(v'\right) = td_{G'}\left(v'\right) + \mu\left(vv'\right) = td_G(v) + \mu\left(vv'\right) = \Delta^t$$

If this happens for every vertices with total degree less than Δ^t in G and their copies in G', then the procedure stops here. If for some vertex v with $td(v) < \Delta^t$, $\mu(vv') = \min\left\{\sigma_G(v), \Delta^t - td_G(v)\right\} = \sigma_G(v)$. Then $td_{G_1}(v) < \Delta^t$ and $td_{G_1}(v) < \Delta^t$. Now, repeat the above procedure for the fuzzy graph G_1 and let the resultant fuzzy graph be G_2. If all the vertices in G_2 have total degree Δ^t, stop.

Otherwise, continue the procedure till the vertices have total degree Δ^t in the resultant fuzzy graph. Let

$$n = \left\lceil \max\left\{\frac{\Delta^t - td_G(v)}{\sigma(v)} / td_G(v) < \Delta^t\right\}\right\rceil.$$

Then the procedure stops after n steps with the Δ^t-totally regular fuzzy graph G_n. Also G is an induced fuzzy subgraph of G_n. Here the number of vertices in

$$\begin{aligned} G_n &= p + p + 2p + 2^2p + \cdots + 2^{n-1}p \\ &= p + p\left(1 + 2 + 2^2 + \cdots + 2^{n-1}\right) \\ &= p + \left(\frac{2^n - 1}{2 - 1}\right)p \\ &= p + (2^n - 1)p \\ &= 2^n p \end{aligned}$$

The number of edges in $G_n = nq + \sum\limits_{v \in V} \left\lceil \frac{\Delta^t - td_G(v)}{\sigma_G(v)} \right\rceil$.

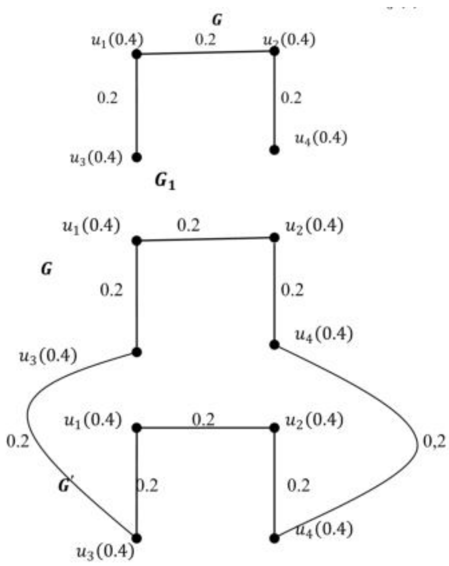

Figure 6.

□

Theorem 4.6. *Let $G : (\sigma, \mu)$ be a fuzzy graph such that both σ and μ are constant partially regular fuzzy graph. Then G is a totally regular fuzzy graph if and only if G is a partially regular fuzzy graph.*

Proof. Assume that G is a k-totally regular fuzzy graph. Let $\mu(uv) = c$ for all $uv \in V$ and $\sigma(u) = c_1$ for all $u \in V$ where c and c_1 are constants. Then

$$
\begin{aligned}
td_G(u) &= d_G(u) + \sigma(u) \\
&= \sum_{u \in v} \mu(uv) + \sigma(u) \\
\Rightarrow k &= c d_G^*(u) + c_1 \\
\Rightarrow d_G^*(u) &= \frac{k - c_1}{c} \quad \text{for all } u \in V.
\end{aligned}
$$

So G^* is regular and hence G is a partially regular fuzzy graph.

Conversely, assume that G is a partially regular fuzzy graph. Let G^* be a r-regular graph. Then

$$
\begin{aligned}
td_G(u) &= d_G(u) + \sigma(u) \\
\Rightarrow td_G(u) &= c d_G^*(u) + c_1 \\
&= cr + c_1 \quad \text{for all } u \in V.
\end{aligned}
$$

So G is totally regular fuzzy graph. □

The following theorem will be helpful in studying various properties of totally regular fuzzy graphs.

Theorem 4.7. *Let $G_1 : (\sigma_1, \mu_1)$ and $G_2 : (\sigma_2, \mu_2)$ be two fuzzy graphs such that $\sigma_1 = \mu_2$. Then $\sigma_1 = \sigma_2$.*

Proof. Since by the definition of a fuzzy graph, $\mu_2(uv) \le \sigma_2(u) \wedge \sigma_2(v)$, for all $u, v \in V_2$. We have $\min \mu_2 \le \sigma_2$. Now $\sigma_1 \le \mu_2 \Rightarrow \sigma_1 \le \min \mu_2$. Therefore $\sigma_1 \le \min \mu_2 \le \sigma_2 \Rightarrow \sigma_1 \le \sigma_2$. □

5. Totally Regular Property of Union of Two Fuzzy Graphs

Theorem 5.1 ([10]). *Let $G_1 : (\sigma_1, \mu_1)$ and $G_2 : (\sigma_2, \mu_2)$ be two fuzzy graphs with underlying crisp graphs (V_1, E_1) and (V_2, E_2) respectively.*

(1). If $u \in V_1 \cup V_2$, and u is arbitrary, then $td_{G_1 \cup G_2}(u) = \begin{cases} td_{G_1}(u), & u \in V_1 \\ td_{G_2}(u), & u \in V_2. \end{cases}$

(2). If $u \in V_1 \cap V_2$ but no edge incident at u lies in $E_1 \cap E_2$. Then any edge incident at u is either in E_1 or in E_2 but not both. Also all these edges will be included in $G_1 \cup G_2$.

$$td_{G_1 \cup G_2}(u) = td_{G_1}(u) + td_{G_2}(u) - \sigma_1(u) \wedge \sigma_2(u)$$

(3). If $u \in V_1 \cap V_2$ and some edges incident at u are in $E_1 \cap E_2$. Any edge uv which is in $E_1 \cap E_2$ appear only once in $G_1 \cup G_2$ and for this uv,

$$td_{G_1 \cup G_2}(u) = td_{G_1}(u) + td_{G_2}(u) - \sigma_1(u) \wedge \sigma_2(u) - \sum_{uv \in E_1 \cap E_2} \mu_1(uv) \wedge \mu_2(uv).$$

Theorem 5.2. *If G_1 and G_2 are two disjoint k-totally regular fuzzy graphs, then $G_1 \cup G_2$ is a k-totally regular fuzzy graph.*

Proof. Since G_1 and G_2 are disjoint fuzzy graphs.

$$td_{G_1 \cup G_2}(u) = \begin{cases} td_{G_1}(u), if\ u \in V_1 \\ td_{G_2}(u), if\ u \in V_2 \end{cases}$$

$$= k, \quad \text{for every } u \in V_1 \cup V_2$$

Therefore $G_1 \cup G_2$ is k-totally regular. □

Remark 5.3. *The above theorem does not hold when G_1 and G_2 are edge disjoint but not vertex disjoint fuzzy graphs.*

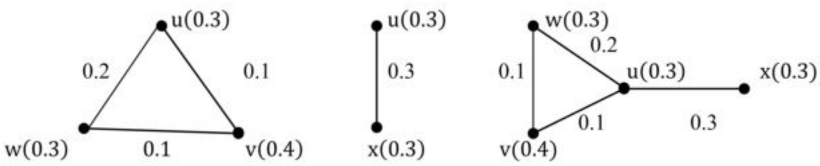

Figure 7.

In the above Figure 7, G_1 and G_2 are 0.6-totally regular fuzzy graph but $G_1 \cup G_2$ is not k-totally regular. Here G_1 and G_2 are edge disjoint but they have the vertex u in common.

6. Conclusion

In this paper, Some properties of total degree and totally regular fuzzy graphs are studied and every fuzzy graph is an induced subgraph of a totally regular fuzzy graph is proved. Also the total degree of a vertex in fuzzy graphs formed by the operation Union in terms of the total degree of vertices in the given fuzzy graphs for some particular cases are obtained. Using them, their totally regular property is studied.

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