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Weakly g-ω-closed Sets

Research Article

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Abstract: In this paper, another generalized class of τ called weakly g-ω-closed sets is studied and the notion of weakly g-ω-open sets in topological spaces is also studied. The relationships of weakly g-ω-closed sets with various other sets are investigated.

MSC: 54A05, 54A10.

Keywords: τ, generalized class, weakly g-ω-closed set, topological space, generalized closed set, g-ω-closed set, preclosed set, preopen set.

1. Introduction

The first step of generalizing closed sets (briefly, g-closed sets) was done by Levine in 1970 [6]. He defined a subset S of a topological space (X, τ) to be g-closed if its closure is contained in every open superset of S. As the weak form of g-closed sets, the notion of weakly g-closed sets was introduced and studied by Sundaram and Nagaveni [11]. Sundaram and Pushpalatha [12] introduced and studied the notion of strongly g-closed sets, which are weaker than closed sets and stronger than g-closed sets. Park and Park [9] introduced and studied the notion of mildly g-closed sets, which is properly placed between the class of strongly g-closed sets and the class of weakly g-closed sets. Moreover, the relations with other notions directly or indirectly connected with g-closed sets were investigated by them. The notion of ω-open sets in topological spaces introduced by Hdeib [4] has been studied in recent years by a good number of researchers like Noiri et al [8], Al-Omari and Noorani [1, 2] and Khalid Y. Al-Zoubi [5]. The main aim of this paper is to study another generalized class of τ called weakly g-ω-open sets in topological spaces. Moreover, this generalized class of τ generalize g-ω-open sets and weakly g-ω-open sets. The relationships of weakly g-ω-closed sets with various other sets are discussed.

2. Preliminaries

Throughout this paper, $\mathbb{R}$ (resp. $\mathbb{Q}$, $(\mathbb{R}-\mathbb{Q})$, $(\mathbb{R}-\mathbb{Q})_-$ and $(\mathbb{R}-\mathbb{Q})_+$) denotes the set of real numbers (resp. the set of rational numbers, the set of irrational numbers, the set of negative irrational numbers and the set of positive irrational numbers). In this paper, (X, τ) represents a topological space on which no separation axioms are assumed unless explicitly stated. The closure and interior of a subset G of a topological space (X, τ) will be denoted by $cl(G)$ and $int(G)$, respectively.

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Definition 2.1. *A subset G of a topological space (X, τ) is said to be*

(1). g-closed [6] if $cl(G) \subseteq H$ whenever $G \subseteq H$ and H is open in X;

(2). g-open [6] if $X \backslash G$ is g-closed;

(3). weakly g-closed [11] if $cl(int(G)) \subseteq H$ whenever $G \subseteq H$ and H is open in X;

(4). strongly g-closed [12] if $cl(G) \subseteq H$ whenever $G \subseteq H$ and H is g-open in X.

Definition 2.2 ([14]). *In a topological space (X, τ), a point p in X is called a condensation point of a subset H if for each open set U containing p, $U \cap H$ is uncountable.*

Definition 2.3 ([4]). *A subset H of a topological space (X, τ) is called ω-closed if it contains all its condensation points. The complement of an ω-closed set is called ω-open.*

It is well known that a subset W of a topological space (X, τ) is ω-open if and only if for each $x \in W$, there exists $U \in \tau$ such that $x \in U$ and $U - W$ is countable. The family of all ω-open sets, denoted by τ_ω, is a topology on X, which is finer than τ. The interior and closure operator in (X, τ_ω) are denoted by int_ω and cl_ω respectively.

Lemma 2.4 ([4]). *Let H be a subset of a topological space (X, τ). Then*

(1). H is ω-closed in X if and only if $H = cl_\omega(H)$.

(2). $cl_\omega(X \backslash H) = X \backslash int_\omega(H)$.

(3). $cl_\omega(H)$ is ω-closed in X.

(4). $x \in cl_\omega(H)$ if and only if $H \cap G \neq \phi$ for each ω-open set G containing x.

(5). $cl_\omega(H) \subseteq cl(H)$.

(6). $int(H) \subseteq int_\omega(H)$.

Lemma 2.5 ([5]). *If A is an ω-open subset of a space (X, τ), then $A - C$ is ω-open for every countable subset C of X.*

Definition 2.6. *A subset G of a topological space (X, τ) is said to be*

(1). preopen [7] if $G \subseteq int(cl(G))$.

(2). preclosed [7] if $X \backslash G$ is preopen (or) $cl(int(G)) \subseteq G$.

(3). g-ω-closed [13] if $cl(G) \subseteq H$ whenever $G \subseteq H$ and H is ω-open in (X, τ).

(4). regular closed [10] if $G = cl(int(G))$.

Definition 2.7 ([3]). *In a topological space (X, τ), a subset G of X is said to be weakly g-ω-closed if $cl(int(G)) \subseteq H$ whenever $G \subseteq H$ and H is ω-open in X.*

Example 2.8 ([3]). *In $\mathbb{R}$ with the topology $\tau = \{\phi, \mathbb{R}, \mathbb{Q}\}$,*

(1). For $G = \mathbb{R} - \mathbb{Q}$, if H is any ω-open subset of $\mathbb{R}$ such that $G \subseteq H$, then $cl(int(G)) = cl(\phi) = \phi \subseteq H$ and hence G is weakly g-ω-closed in X.

(2). $K = \mathbb{Q} \subseteq \mathbb{Q}$, $\mathbb{Q}$ being ω-open whereas $cl(int(\mathbb{Q})) = cl(\mathbb{Q}) = \mathbb{R} \nsubseteq \mathbb{Q}$ which implies $K = \mathbb{Q}$ is not weakly g-ω-closed in X.

Definition 2.9 ([3]). *A subset G in a topological space (X, τ) is said to be weakly g-ω-open if $X \backslash G$ is weakly g-ω-closed.*

Theorem 2.10 ([3]). *In a topological space (X, τ), a subset G of X is weakly g-ω-closed $\Leftrightarrow cl(int(G)) \subseteq G$.*

Proposition 2.11 ([3]). *In a topological space (X, τ), every g-ω-closed set is weakly g-ω-closed but not conversely.*

Theorem 2.12 ([13]). *In a topological space (X, τ), a subset G is closed if and only if it is g-ω-closed.*

Theorem 2.13 ([13]). *In a topological space (X, τ), every g-ω-closed set is g-closed but not conversely.*

3. Properties of Weakly g-ω-closed Sets

Theorem 3.1. *In a topological space (X, τ), for a subset G of X, the following properties are equivalent.*

(1). G is weakly g-ω-closed;

(2). $cl(int(G)) \backslash G = \phi$;

(3). $cl(int(G)) \subseteq G$;

(4). G is preclosed.

Proof.
(1) $\Leftrightarrow$ (2) G is weakly g-ω-closed $\Leftrightarrow cl(int(G)) \subseteq G$ by Theorem 2.10 $\Leftrightarrow cl(int(G)) \backslash G = \phi$.
(2) $\Leftrightarrow$ (3) $cl(int(G)) \backslash G = \phi \Leftrightarrow cl(int(G)) \subseteq G$.
(3) $\Leftrightarrow$ (4) $cl(int(G)) \subseteq G \Leftrightarrow G$ is preclosed by (2) of Definition 2.6. □

Theorem 3.2. *In a topological space (X, τ), if G is weakly g-ω-closed, then $G \cup (X - cl(int(G)))$ is weakly g-ω-closed.*

Proof. Since G is weakly g-ω-closed, $cl(int(G)) \subseteq G$ by Theorem 2.10. Then $X - G \subseteq X - cl(int(G))$ and $G \cup (X - G) \subseteq G \cup (X - cl(int(G)))$. Thus $X \subseteq G \cup (X - cl(int(G)))$ and so $G \cup (X - cl(int(G))) = X$. Hence $G \cup (X - cl(int(G)))$ is weakly g-ω-closed. □

Theorem 3.3. *In a topological space (X, τ), the following properties are equivalent:*

(1). G is a closed and an open set,

(2). G is a regular closed and an open set,

(3). G is a weakly g-ω-closed and an open set.

Proof.
(1) $\Rightarrow$ (2): Since G is closed and open, $G = cl(G)$ and $G = int(G)$. Thus $G = cl(int(G))$ and $G = int(G)$. Hence G is regular closed and open.
(2) $\Rightarrow$ (3): Since G is regular closed and open, $G = cl(int(G))$. Thus $cl(int(G)) \subseteq G$. By Theorem 2.10, G is weakly g-ω-closed and open.
(3) $\Rightarrow$ (1): Since G is weakly g-ω-closed, $cl(int(G)) \subseteq G$ by Theorem 2.10. Again G is open implies $cl(G) = cl(int(G)) \subseteq G$. Thus G is closed and open. □

Theorem 3.4. *In a topological space (X, τ), every closed set is weakly g-ω-closed.*

Proof. If A is closed, then A is g-ω-closed by Theorem 2.12. By Proposition 2.11, A is weakly g-ω-closed. □

Remark 3.5. *The converse of Theorem 3.4 is not true follows from the following example.*

Example 3.6. *In $\mathbb{R}$ with the topology $\tau = \{\phi, \mathbb{R}, \{1\}\}$, for $G = \mathbb{R} - \mathbb{Q}$, if H is any ω-open subset of $\mathbb{R}$ such that $G \subseteq H$, then $cl(int(G)) = cl(\phi) = \phi \subseteq H$ and hence G is weakly g-ω-closed. But G is not closed for $cl(G) = \mathbb{R} - \{1\} \nsubseteq G$.*

Remark 3.7. *The following example shows that the concepts of g-closedness and weakly g-ω-closedness are independent of each other.*

Example 3.8. *In $\mathbb{R}$ with the topology $\tau = \{\phi, \mathbb{R}, \mathbb{R} - \mathbb{Q}\}$,*

(1). for $G = (\mathbb{R} - \mathbb{Q})_+ =$ the set of positive irrationals, $int(G) = \phi$. So $cl(int(G)) = cl(\phi) = \phi \subseteq G$ and thus G is weakly g-ω-closed by Theorem 2.10. But G is not g-closed for $G \subseteq (\mathbb{R} - \mathbb{Q}) \in \tau$ whereas $cl(G) = \mathbb{R} \nsubseteq \mathbb{R} - \mathbb{Q}$.

(2). for $H = (\mathbb{R} - \mathbb{Q}) \cup \{1\}$, $\mathbb{R}$ is the only open set containing H. Hence H is g-closed. But H is not weakly g-ω-closed for $cl(int(H)) = cl(\mathbb{R} - \mathbb{Q}) = \mathbb{R} \nsubseteq H$.

Remark 3.9. *In a topological space (X, τ), the following relations hold for a subset G of X.*

$$\textit{closed} \leftrightarrow g\text{-}\omega\text{-}\textit{closed} \rightarrow \textit{weakly } g\text{-}\omega\text{-}\textit{closed} \not\leftrightarrow g\text{-}\textit{closed}$$

Where $A \longleftrightarrow B$ means A implies and is implied by B, $A \rightarrow B$ means A implies B but not conversely and $A \not\leftrightarrow B$ means A and B are independent.

Theorem 3.10. *In a topological space (X, τ), $cl(A)$ is always weakly g-ω-closed for every subset A of X.*

Proof. Since $cl(cl(A)) \subseteq cl(A)$, $cl(A)$ is closed. Hence $cl(A)$ is g-ω-closed by Theorem 2.12 and weakly g-ω-closed by Proposition 2.11. □

4. Further Properties

Theorem 4.1. *In a topological space (X, τ), if G is weakly g-ω-closed and H is a subset such that $G \subseteq H \subseteq cl(int(G))$, then H is weakly g-ω-closed.*

Proof. Since G is weakly g-ω-closed, $cl(int(G)) \subseteq G$ by (3) of Theorem 3.1. Thus by assumption, $G \subseteq H \subseteq cl(int(G)) \subseteq G$. Then $G = H$ and so H is weakly g-ω-closed. □

Corollary 4.2. *Let (X, τ) be a topological space. If G is a weakly g-ω-closed set and an open set, then $cl(G)$ is weakly g-ω-closed.*

Proof. Since G is open in X, $G \subseteq cl(G) \subseteq cl(G) = cl(int(G))$. G is weakly g-ω-closed implies $cl(G)$ is weakly g-ω-closed by Theorem 4.1. □

Theorem 4.3. *In a topological space (X, τ), a nowhere dense subset is weakly g-ω-closed.*

Proof. If G is a nowhere dense subset in X then $int(cl(G)) = \phi$. Since $int(G) \subseteq int(cl(G))$, $int(G) = \phi$. Hence $cl(int(G)) = cl(\phi) = \phi \subseteq G$. Thus, G is weakly g-ω-closed in (X, τ) by Theorem 2.10. □

Remark 4.4. *The converse of Theorem 4.3 is not true in general as shown in the following example.*

Example 4.5. *In Example 3.8, $G = (\mathbb{R} - \mathbb{Q})_+$ is weakly g-ω-closed. On the other hand, $int(cl(G)) = int(\mathbb{R}) = \mathbb{R} \neq \phi$ and thus $G = (\mathbb{R} - \mathbb{Q})_+$ is not nowhere dense in X.*

Remark 4.6. *In a topological space (X,τ), the intersection of two weakly g-ω-closed subsets is weakly g-ω-closed.*

Proof. Let A and B be weakly g-ω-closed subsets in (X,τ). Then $cl(int(A)) \subseteq A$ and $cl(int(B)) \subseteq B$ by Theorem 2.10. Also $cl[int(A\cap B)] \subseteq cl[int(A)] \cap cl[int(B)] \subseteq A\cap B$. This implies that $A\cap B$ is weakly g-ω-closed by Theorem 2.10. □

Remark 4.7. *In a topological space (X,τ), the union of two weakly g-ω-closed subsets need not be weakly g-ω-closed.*

Example 4.8. *In Example 3.8, for $A = (\mathbb{R}-\mathbb{Q})_+ =$ the set of positive irrationals and $B = (\mathbb{R}-\mathbb{Q})_- =$ the set of negative irrationals, $int(A) = \phi$ and $int(B) = \phi$ respectively. So $cl(int(A)) = cl(\phi) = \phi \subseteq A$ and thus A is weakly g-ω-closed by Theorem 2.10. Similarly B is also weakly g-ω-closed. But $int(A\cup B) = int(\mathbb{R}-\mathbb{Q}) = \mathbb{R}-\mathbb{Q}$. So $cl[int(A\cup B)] = cl(\mathbb{R}-\mathbb{Q}) = \mathbb{R} \nsubseteq \mathbb{R}-\mathbb{Q} = A\cup B$. Hence $A\cup B$ is not weakly g-ω-closed.*

Theorem 4.9. *In a topological space (X,τ), a subset G is weakly g-ω-open if and only if $G \subseteq int(cl(G))$.*

Proof. G is weakly g-ω-open $\Leftrightarrow X\backslash G$ is weakly g-ω-closed $\Leftrightarrow X\backslash G$ is preclosed by (4) of Theorem 3.1 $\Leftrightarrow G$ is preopen $\Leftrightarrow G \subseteq int(cl(G))$. □

Theorem 4.10. *In a topological space (X,τ), if the subset G is weakly g-ω-closed, then $cl(int(G))\backslash G$ is weakly g-ω-open in (X,τ).*

Proof. Since G is weakly g-ω-closed, $cl(int(G))\backslash G = \phi$ by (2) of Theorem 3.1. Thus $cl(int(G))\backslash G$ is weakly g-ω-open in (X,τ). □

Theorem 4.11. *In a topological space (X,τ), if G is weakly g-ω-open, then $int(cl(G)) \cup (X-G) = X$.*

Proof. Since G is weakly g-ω-open, $G \subseteq int(cl(G))$ by Theorem 4.9. So $(X-G)\cup G \subseteq (X-G)\cup int(cl(G))$ which implies $X = (X-G)\cup int(cl(G))$. □

Theorem 4.12. *In a topological space (X,τ), if G is weakly g-ω-open and H is a subset such that $int(cl(G)) \subseteq H \subseteq G$, then H is weakly g-ω-open.*

Proof. Since G is weakly g-ω-open, $G \subseteq int(cl(G))$ by Theorem 4.9. By assumption $int(cl(G)) \subseteq H \subseteq G$. This implies $G \subseteq int(cl(G)) \subseteq H \subseteq G$. Thus $G = H$ and so H is weakly g-ω-open. □

Corollary 4.13. *In a topological space (X,τ), if G is a weakly g-ω-open set and a closed set, then $int(G)$ is weakly g-ω-open.*

Proof. Let G be a weakly g-ω-open set and a closed set in (X,τ). Then $int(cl(G)) = int(G) \subseteq int(G) \subseteq G$. Thus, by Theorem 4.12, $int(G)$ is weakly g-ω-open in (X,τ). □

Definition 4.14. *A subset H of a topological space (X,τ) is called a W-set if $H = M\cup N$ where M is ω-closed and N is preopen.*

Proposition 4.15. *Every preopen (resp. ω-closed) set is a W-set.*

Remark 4.16. *The separate converses of Proposition 4.15 are not true in general as shown in the following example.*

Example 4.17.

(1). Let $\mathbb{R}$ and τ be as in Example 2.8 and $G = \mathbb{R}-\mathbb{Q}$. Since G is closed, it is ω-closed and hence a W-set. But $int(cl(G)) = int(G) = \phi \nsupseteq G$. Hence $G = \mathbb{R}-\mathbb{Q}$ is not preopen.

(2). In Example 3.8, for $G = \mathbb{R} - \mathbb{Q}$, $int(cl(G)) = int(\mathbb{R}) = \mathbb{R} \supseteq G$. Thus G is preopen and hence a W-set. But G is not ω-closed for any $x \in \mathbb{Q}$ is a condensation point of G and $x \notin G$.

Remark 4.18. *The following example shows that the concepts of preopenness and ω-closedness are independent of each other.*

Example 4.19. *In Example 4.17(1), $G = \mathbb{R} - \mathbb{Q}$ is ω-closed but not preopen. In Example 4.17(2), $G = \mathbb{R} - \mathbb{Q}$ is preopen but not ω-closed.*

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