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Strongly g-$\star$-closed Sets

Research Article

R.Umamaheswari[1], R.Premkumar[1] and O.Ravi[1*]

1 Department of Mathematics, P.M.Thevar College, Usilampatti, Madurai, Tamil Nadu, India.

Abstract: In this paper, the notion of strongly g-$\star$-closed sets is introduced in ideal topological spaces. Characterizations and properties of strongly g-$\star$-closed sets and strongly g-$\star$-open sets are given. A characterization of normal spaces is given in terms of strongly g-$\star$-open sets. Also it is established that a strongly g-$\star$-closed subset of an $\mathcal{I}$-compact space is $\mathcal{I}$-compact.

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1. Introduction and Preliminaries

By a space, we always mean a topological space (X, τ) with no separation properties assumed. If H⊆X, cl(H) and int(H) will, respectively, denote the closure and interior of H in (X, τ).

Example 1.1. *A subset H of a space (X, τ) is called semi-open [8] if H⊆cl(int(H)).*

Definition 1.2. *A subset H of a space (X, τ) is said to be g-closed [9] if cl(H)⊆U whenever H⊆U and U is open in X.*

An ideal $\mathcal{I}$ on a space (X, τ) is a nonempty collection of subsets of X which satisfies (i) A∈$\mathcal{I}$ and B⊆A⇒B∈$\mathcal{I}$ and (ii) A∈$\mathcal{I}$ and B∈$\mathcal{I}$⇒A∪B∈$\mathcal{I}$ [7]. Given a space (X, τ) with an ideal $\mathcal{I}$ on X and if $\wp$(X) is the set of all subsets of X, a set operator $(.)^*$: $\wp$(X)→$\wp$(X), called a local function [7] of A with respect to τ and $\mathcal{I}$, is defined as follows: for A⊆X, $A^*(\mathcal{I},\tau)$={x∈X | U∩A∉$\mathcal{I}$ for every U∈τ(x)} where τ(x)={U∈τ | x∈U}. We will make use of the basic facts about the local functions [[6], Theorem 2.3] without mentioning it explicitly. A Kuratowski closure operator $cl^*(.)$ for a topology $\tau^*(\mathcal{I},\tau)$, called the $\star$-topology, finer than τ is defined by $cl^*(A)=A\cup A^*(\mathcal{I},\tau)$ [14]. When there is no chance for confusion, we will simply write A^* for $A^*(\mathcal{I},\tau)$ and τ^* for $\tau^*(\mathcal{I},\tau)$. If $\mathcal{I}$ is an ideal on X, then (X, τ, $\mathcal{I}$) is called an ideal topological space.

Lemma 1.3 ([6]). *Let (X, τ, $\mathcal{I}$) be an ideal topological space and A, B subsets of X. Then the following properties hold:*

(1) $A\subseteq B\Rightarrow A^*\subseteq B^*$,

(2) $A^*=cl(A^*)\subseteq cl(A)$,

(3) $(A^*)^*\subseteq A^*$,

(4) $(A\cup B)^*=A^*\cup B^*$,

* E-mail: siingam@yahoo.com

(5) $(A\cap B)^*\subseteq A^*\cap B^*$.

Definition 1.4. *A subset H of an ideal topological space* $(X, \tau, \mathcal{I})$ *is called* $\star$*-closed [6] (resp.* $\star$*-dense in itself [5]) if* $H^*\subseteq H$ *(resp.* $H\subseteq H^*$*). The complement of a* $\star$*-closed set is called* $\star$*-open.*

Definition 1.5. *A subset H of an ideal topological space* $(X, \tau, \mathcal{I})$ *is called* $\mathcal{I}_g$*-closed [2, 11] if* $H^*\subseteq U$ *whenever* $H\subseteq U$ *and U is open in* $(X, \tau, \mathcal{I})$.

Definition 1.6 ([2]). *An ideal topological space* $(X, \tau, \mathcal{I})$ *is called* $T_{\mathcal{I}}$ *if every* $\mathcal{I}_g$*-closed subset of X is* $\star$*-closed in X.*

Lemma 1.7. *If* $(X, \tau, \mathcal{I})$ *is a* $T_{\mathcal{I}}$ *space and* $A\subseteq X$ *is an* $\mathcal{I}_g$*-closed set, then A is a* $\star$*-closed set [[11], Corollary 2.2].*

Lemma 1.8. *In an ideal topological space* $(X,\tau,\mathcal{I})$*, every g-closed set is* $\mathcal{I}_g$*-closed but not conversely [[2], Theorem 2.1].*

Definition 1.9 ([10]). *A subset H of an ideal topological space* $(X, \tau, \mathcal{I})$ *is said to be*

(1) $\star$*-g-closed if* $cl(H)\subseteq U$ *whenever* $H\subseteq U$ *and U is* $\star$*-open in* $(X, \tau, \mathcal{I})$*,*

(2) $\star$*-g-open if its complement is* $\star$*-g-closed.*

Recall that every open set is $\star$*-g-open but not conversely.*

Proposition 1.10 ([1]). *If A is* $\star$*-g-closed of* $(X, \tau, \mathcal{I})$ *and B is closed in X, then* $A \cap B$ *is* $\star$*-g-closed in* $(X, \tau, \mathcal{I})$.

Definition 1.11 ([1]). *A subset A of an ideal topological space* $(X,\tau,\mathcal{I})$ *is said to be*

(1) strongly $\mathcal{I}_g$*-*$\star$*-closed if* $A^* \subseteq U$ *whenever* $A \subseteq U$ *and U is* $\star$*-g-open in* $(X, \tau, \mathcal{I})$.

(2) strongly $\mathcal{I}_g$*-*$\star$*-open if its complement is strongly* $\mathcal{I}_g$*-*$\star$*-closed.*

Theorem 1.12 ([1]). *In an ideal topological space* $(X, \tau, \mathcal{I})$*, for* $A\subseteq X$*, the following statements are equivalent.*

(1) A is strongly $\mathcal{I}_g$*-*$\star$*-closed,*

(2) $cl^*(A)\subseteq U$ *whenever* $A\subseteq U$ *and U is* $\star$*-g-open in X,*

(3) $cl^*(A)-A$ *contains no nonempty* $\star$*-g-closed set,*

(4) A^*-A *contains no nonempty* $\star$*-g-closed set.*

Theorem 1.13 ([1]). *Let* $(X, \tau, \mathcal{I})$ *be an ideal topological space. If A and B are subsets of X such that* $A\subseteq B\subseteq cl^*(A)$ *and A is strongly* $\mathcal{I}_g$*-*$\star$*-closed, then B is strongly* $\mathcal{I}_g$*-*$\star$*-closed.*

Definition 1.14. *An ideal* $\mathcal{I}$ *is said to be codense [3] or* τ*-boundary [12] if* $\tau \cap \mathcal{I}=\{\phi\}$.

Theorem 1.15 ([1]). *In an ideal topological space* $(X,\tau,\mathcal{I})$*, every* $\star$*-closed set is strongly* $\mathcal{I}_g$*-*$\star$*-closed but not conversely.*

Theorem 1.16 ([1]). *In an ideal topological space* $(X,\tau,\mathcal{I})$*, every strongly* $\mathcal{I}_g$*-*$\star$*-closed set is* $\mathcal{I}_g$*-closed but not conversely.*

Lemma 1.17. *Let* $(X, \tau, \mathcal{I})$ *be an ideal topological space. Then* $\mathcal{I}$ *is codense if and only if* $G\subseteq G^*$ *for every semi-open set G in X [[13], Theorem 3].*

Lemma 1.18. *Let* $(X, \tau, \mathcal{I})$ *be an ideal topological space and* $A\subseteq X$*. If* $A\subseteq A^*$*, then* $A^*=cl(A^*)=cl(A)=cl^*(A)$ *[[13], Theorem 5].*

Definition 1.19. *A subset H of an ideal topological space* $(X, \tau, \mathcal{I})$ *is said to be* $\mathcal{I}$*-compact [4] or compact modulo* $\mathcal{I}$ *[12] if for every open cover* $\{U_\alpha \mid \alpha\in\Delta\}$ *of H, there exists a finite subset* Δ_0 *of* Δ *such that* $H-\cup\{U_\alpha \mid \alpha\in\Delta_0\}\in\mathcal{I}$. *The space (X,* τ, $\mathcal{I}$*) is* $\mathcal{I}$*-compact if X is* $\mathcal{I}$*-compact as a subset.*

Theorem 1.20. *Let (X,* τ, $\mathcal{I}$*) be an ideal topological space. If A is an* $\mathcal{I}_g$*-closed subset of X, then A is* $\mathcal{I}$*-compact [[11], Theorem 2.17].*

2. Properties of Strongly g-$\star$-closed Sets

Definition 2.1. *A subset A of an ideal topological space (X,* τ, $\mathcal{I}$*) is said to be*

(1) strongly g-$\star$*-closed if* $cl(A) \subseteq U$ *whenever* $A \subseteq U$ *and U is* $\star$-g*-open in X,*

(2) strongly g-$\star$*-open if its complement is strongly* g-$\star$*-closed.*

Theorem 2.2. *In an ideal topological space (X,* τ, $\mathcal{I}$*), every strongly* g-$\star$*-closed set is g-closed.*

Proof. It follows from the fact that every open set is $\star$-g-open. □

The converses of Theorem 2.2 is not true in general as shown in the following example.

Example 2.3. *Let* $X=\{a, b, c\}$, $\tau=\{\phi, X, \{a\}, \{b, c\}\}$ *and* $\mathcal{I}=\{\phi, \{a\}\}$. *Then strongly* g-$\star$*-closed sets are* ϕ, X, $\{a\}$, $\{b, c\}$ *and g-closed sets are* ϕ, X, $\{a\}$, $\{b\}$, $\{c\}$, $\{a, b\}$, $\{a, c\}$, $\{b, c\}$. *Clearly* $\{b\}$ *is g-closed but not strongly* g-$\star$*-closed.*

The following Theorem gives characterizations of strongly g-$\star$-closed sets.

Theorem 2.4. *In an ideal topological space (X,* τ, $\mathcal{I}$*), for* $A\subseteq X$*, the following statements are equivalent.*

(1) A is strongly g-$\star$*-closed,*

(2) cl(A) $\subseteq$ *U whenever A* $\subseteq$ *U and U is* $\star$-g*-open in X,*

(3) cl(A) $-$ *A contains no nonempty* $\star$-g*-closed set.*

Proof.

(1) ⇒ (2) Let A ⊆ U where U is $\star$-g-open in X. Since A is strongly g-$\star$-closed, cl(A) ⊆ U.

(2) ⇒ (3) Let F be a $\star$-g-closed subset such that F ⊆ cl(A) − A. Then F ⊆ cl(A). Also F ⊆ cl(A) − A ⊆ X − A and hence A ⊆ X − F where X − F is $\star$-g-open. By (2) cl(A) ⊆ X − F and so F ⊆ X − cl(A). Thus F ⊆ cl(A) ∩ (X − cl(A)) = ϕ.

(3) ⇒ (1) Let A ⊆ U where U is $\star$-g-open in X. Then X − U ⊆ X − A and so cl(A) ∩ (X − U) ⊆ cl(A) ∩ (X − A) = cl(A) − A. Since cl(A) is always a closed subset and X − U is $\star$-g-closed, cl(A) ∩ (X − U) is a $\star$-g-closed set contained in cl(A) − A and hence cl(A) ∩ (X − U) = ϕ by (3). Thus cl(A) ⊆ U and A is strongly g-$\star$-closed. □

Theorem 2.5. *Every closed set is strongly* g-$\star$*-closed.*

Proof. Let A be closed. To prove A is strongly g-$\star$-closed, let U be any $\star$-g-open subset such that A ⊆ U. Since A is closed, cl(A) ⊆ A ⊆ U. Thus A is strongly g-$\star$-closed. □

The converse of Theorem 2.5 is not true in general as shown in the following example.

Example 2.6. *Let* $X=\{a, b, c, d\}$, $\tau=\{\phi, X, \{d\}, \{a, c\}, \{a, c, d\}\}$ *and* $\mathcal{I}=\{\phi, \{a\}, \{d\}, \{a, d\}\}$. *Then strongly* g-$\star$*-closed sets are* ϕ, X, $\{b\}$, $\{a, b\}$, $\{b, c\}$, $\{b, d\}$, $\{a, b, c\}$, $\{a, b, d\}$, $\{b, c, d\}$ *and closed sets are* ϕ, X, $\{b\}$, $\{b, d\}$, $\{a, b, c\}$. *Clearly* $\{b, c\}$ *is strongly* g-$\star$*-closed but not closed.*

Theorem 2.7. *In an ideal topological space $(X, \tau, \mathcal{I})$, A^* is always strongly g-$\star$-closed for every subset A of X.*

Proof. Let $A^* \subseteq U$ where U is $\star$-g-open in X. Since A^* is closed, $cl(A^*) \subseteq A^* \subseteq U$. Hence A^* is strongly g-$\star$-closed. □

Theorem 2.8. *Let $(X, \tau, \mathcal{I})$ be an ideal topological space. Then every strongly g-$\star$-closed, $\star$-g-open set is closed.*

Proof. Let A be strongly g-$\star$-closed and $\star$-g-open. We have $A \subseteq A$ where A is $\star$-g-open. Since A is strongly g-$\star$-closed, $cl(A) \subseteq A$. Thus A is closed. □

Corollary 2.9. *If $(X, \tau, \mathcal{I})$ is a $T_{\mathcal{I}}$ space and A is a strongly g-$\star$-closed set, then A is $\star$-closed set.*

Proof. By assumption A is strongly g-$\star$-closed in $(X, \tau, \mathcal{I})$ and so by Theorem 2.2, A is g-closed and hence $\mathcal{I}_g$-closed by Lemma 1.8. Since $(X, \tau, \mathcal{I})$ is a $T_{\mathcal{I}}$-space, by Definition 1.6, A is $\star$-closed. □

Corollary 2.10. *Let A be a strongly g-$\star$-closed set in $(X, \tau, \mathcal{I})$. Then the following are equivalent.*

(1) A is a closed set,

(2) $cl(A) - A$ is a $\star$-g-closed set.

Proof.

(1) ⇒ (2) By (1) A is closed. Hence $cl(A) \subseteq A$ and $cl(A) - A = \phi$ which is a $\star$-g-closed set.

(2) ⇒ (1) Since A is strongly g-$\star$-closed, by Theorem 2.4(3), $cl(A) - A$ contains no non-empty $\star$-g-closed set. By assumption (2), $cl(A) - A$ is $\star$-g-closed and hence $cl(A) - A = \phi$. Thus $cl(A) \subseteq A$ and hence A is closed. □

Theorem 2.11. *In an ideal topological space $(X, \tau, \mathcal{I})$, every strongly g-$\star$-closed set is strongly $\mathcal{I}_g$-$\star$-closed.*

Proof. Let A be a strongly g-$\star$-closed set. Let U be any $\star$-g-open set such that $A \subseteq U$. Since A is strongly g-$\star$-closed, $cl(A) \subseteq U$. So, $A^* \subseteq A \cup A^* = cl^*(A) \subseteq cl(A) \subseteq U$ and thus A is strongly $\mathcal{I}_g$-$\star$-closed. □

The converse of Theorem 2.11 is not true in general as shown in the following example.

Example 2.12. *In Example 2.6, strongly $\mathcal{I}_g$-$\star$-closed sets are ϕ, X, $\{a\}$, $\{b\}$, $\{d\}$, $\{a, b\}$, $\{a, d\}$, $\{b, c\}$, $\{b, d\}$, $\{a, b, c\}$, $\{a, b, d\}$, $\{b, c, d\}$. Clearly $\{a\}$ is strongly $\mathcal{I}_g$-$\star$-closed but not strongly g-$\star$-closed.*

Theorem 2.13. *If $(X, \tau, \mathcal{I})$ is an ideal topological space and A is a $\star$-dense in itself, strongly $\mathcal{I}_g$-$\star$-closed subset of X, then A is strongly g-$\star$-closed.*

Proof. Let $A \subseteq U$ where U is $\star$-g-open in X. Since A is strongly $\mathcal{I}_g$-$\star$-closed, $A^* \subseteq U$. As A is $\star$-dense in itself, by Lemma 1.18, $cl(A) = A^*$. Thus $cl(A) \subseteq U$ and hence A is strongly g-$\star$-closed. □

Corollary 2.14. *If $(X, \tau, \mathcal{I})$ is any ideal topological space where $\mathcal{I}=\{\phi\}$, then A is strongly $\mathcal{I}_g$-$\star$-closed in X if and only if A is strongly g-$\star$-closed in X.*

Proof. In $(X, \tau, \mathcal{I})$, if $\mathcal{I} = \{\phi\}$ then $A^* = cl(A)$ for the subset A. A is strongly $\mathcal{I}_g$-$\star$-closed in X ⇔ $A^* \subseteq U$ whenever $A \subseteq U$ and U is $\star$-g-open in X ⇔ $cl(A) \subseteq U$ whenever $A \subseteq U$ and U is $\star$-g-open in X ⇔ A is strongly g-$\star$-closed in X. □

Corollary 2.15. *In an ideal topological space $(X, \tau, \mathcal{I})$ where $\mathcal{I}$ is codense, if A is a semi-open and strongly $\mathcal{I}_g$-$\star$-closed subset of X, then A is strongly g-$\star$-closed.*

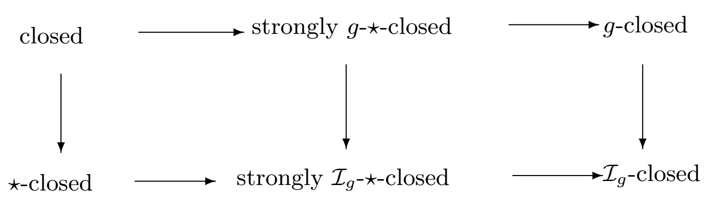
Proof. By Lemma 1.17, A is $\star$-dense in itself. By Theorem 2.13, A is strongly g-$\star$-closed. □

Example 2.16. *In Example 2.3, strongly $\mathcal{I}_g$-$\star$-closed sets are ϕ, X, $\{a\}$, $\{b, c\}$ and g-closed sets are ϕ, X, $\{a\}$, $\{b\}$, $\{c\}$, $\{a, b\}$, $\{a, c\}$, $\{b, c\}$. Clearly $\{b\}$ is g-closed but not strongly $\mathcal{I}_g$-$\star$-closed.*

Example 2.17. *In Example 2.6, g-closed sets are ϕ, X, $\{b\}$, $\{a, b\}$, $\{b, c\}$, $\{b, d\}$, $\{a, b, c\}$, $\{a, b, d\}$, $\{b, c, d\}$. Clearly $\{a\}$ is strongly $\mathcal{I}_g$-$\star$-closed but not g-closed.*

Remark 2.18. *We see that from Examples 2.16 and 2.17, g-closed sets and strongly $\mathcal{I}_g$-$\star$-closed sets are independent.*

Remark 2.19. *We have the following implications for the subsets stated above.*



Theorem 2.20. *Let (X, τ, $\mathcal{I}$) be an ideal topological space and A$\subseteq$X. Then A is strongly g-$\star$-closed if and only if A=F$-$N where F is closed and N contains no nonempty $\star$-g-closed set.*

Proof. If A is strongly g-$\star$-closed, then by Theorem 2.4 (3), N=cl(A) $-$ A contains no nonempty $\star$-g-closed set. If F=cl(A), then F is closed such that F$-$N = (A$\cup$cl(A))-(cl(A)$-$A)=(A$\cup$cl(A))$\cap$(cl(A)$\cap A^c)^c$=(A$\cup$cl(A))$\cap$((cl(A))^c $\cup$ A)=(A$\cup$cl(A))$\cap$(A$\cup$(cl(A))^c)=A$\cup$(cl(A)$\cap$(cl(A))^c)=A.
Conversely, suppose A = F$-$N where F is closed and N contains no nonempty $\star$-g-closed set. Let U be an $\star$-g-open set such that A$\subseteq$U. Then F$-$N$\subseteq$U which implies that F$\cap$(X$-$U)$\subseteq$N. Now A$\subseteq$F and cl(F)$\subseteq$F then cl(A)$\subseteq$cl(F) and so cl(A)$\cap$(X$-$U)$\subseteq$cl(F)$\cap$ (X$-$U)$\subseteq$F$\cap$(X$-$U)$\subseteq$N. Since cl(A)$\cap$(X$-$U) is $\star$-g-closed, by hypothesis cl(A)$\cap$(X$-$U) = ϕ and so cl(A)$\subseteq$U. Hence A is strongly g-$\star$-closed. □

Theorem 2.21. *Let (X, τ, $\mathcal{I}$) be an ideal topological space. If A and B are subsets of X such that A$\subseteq$B$\subseteq$cl(A) and A is strongly g-$\star$-closed, then B is strongly g-$\star$-closed.*

Proof. Since A is strongly g-$\star$-closed, then by Theorem 2.4(3), cl(A)$-$A contains no nonempty $\star$-g-closed set. But cl(B)$-$B$\subseteq$cl(A)$-$A and so cl(B)$-$B contains no nonempty $\star$-g-closed set. Hence B is strongly g-$\star$-closed. □

Corollary 2.22. *Let (X, τ, $\mathcal{I}$) be an ideal topological space. If A and B are subsets of X such that A$\subseteq$B$\subseteq A^*$ and A is strongly $\mathcal{I}_g$-$\star$-closed, then A and B are strongly g-$\star$-closed sets.*

Proof. Let A and B be subsets of X such that A$\subseteq$B$\subseteq$A*. Then A$\subseteq$B$\subseteq$A$^*\subseteq$A$^*\cup$A = cl*(A). Since A is strongly $\mathcal{I}_g$-$\star$-closed, by Theorem 1.13, B is strongly $\mathcal{I}_g$-$\star$-closed. Since A$\subseteq$B$\subseteq$A*, we have A* = B*. Hence A $\subseteq$ A* and B $\subseteq$ B*. Thus A is $\star$-dense in itself and B is $\star$-dense in itself and by Theorem 2.13, A and B are strongly g-$\star$-closed. □

The following Theorem gives a characterization of strongly g-$\star$-open sets.

Theorem 2.23. *Let (X, τ, $\mathcal{I}$) be an ideal topological space and A$\subseteq$X. Then A is strongly g-$\star$-open if and only if F$\subseteq$int(A) whenever F is $\star$-g-closed and F$\subseteq$A.*

Proof. Suppose A is strongly g-$\star$-open. If F is $\star$-g-closed and F$\subseteq$A, then X$-$A$\subseteq$X$-$F and so cl(X$-$A)$\subseteq$X$-$F by Theorem 2.4(2). Therefore F$\subseteq$X$-$cl(X$-$A)=int(A). Hence F$\subseteq$int(A).
Conversely, suppose the condition holds. Let U be an $\star$-g-open set such that X$-$A$\subseteq$U. Then X$-$U$\subseteq$A and so X$-$U$\subseteq$int(A). Therefore cl(X$-$A)$\subseteq$U. By Theorem 2.4(2), X$-$A is strongly g-$\star$-closed. Hence A is strongly g-$\star$-open. □

Corollary 2.24. *Let $(X, \tau, \mathcal{I})$ be an ideal topological space and $A \subseteq X$. If A is strongly g-$\star$-open, then $F \subseteq int(A)$ whenever F is closed and $F \subseteq A$.*

The following Theorem gives a property of strongly g-$\star$-closed.

Theorem 2.25. *Let $(X, \tau, \mathcal{I})$ be an ideal topological space and $A, B \subseteq X$. If A is strongly g-$\star$-open and $int(A) \subseteq B \subseteq A$, then B is strongly g-$\star$-open.*

Proof. Since $\text{int}(A) \subseteq B \subseteq A$, we have $X - A \subseteq X - B \subseteq X - \text{int}(A) = \text{cl}(X - A)$. By assumption A is strongly g-$\star$-open and so $X - A$ is strongly g-$\star$-closed. Hence by Theorem 2.21, $X - B$ is strongly g-$\star$-closed and B is strongly g-$\star$-open. □

The following Theorem gives a characterization of strongly g-$\star$-closed sets in terms of strongly g-$\star$-open sets.

Theorem 2.26. *Let $(X, \tau, \mathcal{I})$ be an ideal topological space and $A \subseteq X$. Then the following are equivalent.*

(1) A is strongly g-$\star$-closed,

(2) $A \cup (X - cl(A))$ is strongly g-$\star$-closed,

(3) $cl(A) - A$ is strongly g-$\star$-open.

Proof.
(1) $\Rightarrow$ (2). Suppose A is strongly g-$\star$-closed. If U is any $\star$-g-open set such that $(A \cup (X - \text{cl}(A))) \subseteq U$, then $X - U \subseteq X - (A \cup (X - \text{cl}(A))) = [A \cup (\text{cl}(A))^c]^c = \text{cl}(A) \cap A^c = \text{cl}(A) - A$. Since A is strongly g-$\star$-closed, by Theorem 2.4(3), it follows that $X - U = \phi$ and so $X = U$. Since X is the only $\star$-g-open set containing $A \cup (X - \text{cl}(A))$, clearly, $A \cup (X - \text{cl}(A))$ is strongly g-$\star$-closed.
(2) $\Rightarrow$ (1). Suppose $A \cup (X - \text{cl}(A))$ is strongly g-$\star$-closed. If F is any $\star$-g-closed set such that $F \subseteq \text{cl}(A) - A = X - (A \cup (X - \text{cl}(A)))$, then $A \cup (X - \text{cl}(A)) \subseteq X - F$ and $X - F$ is $\star$-g-open. Therefore, $\text{cl}(A \cup (X - \text{cl}(A))) \subseteq X - F$ which implies that $\text{cl}(A) \subseteq \text{cl}(A) \cup \text{cl}(X - \text{cl}(A)) = \text{cl}(A \cup (X - \text{cl}(A))) \subseteq X - F$ and so $F \subseteq X - \text{cl}(A)$. Since $F \subseteq \text{cl}(A)$, it follows that $F = \phi$. Hence A is strongly g-$\star$-closed by Theorem 2.4(3).
The equivalence of (2) and (3) follows from the fact that $X - (\text{cl}(A) - A) = A \cup (X - \text{cl}(A))$. □

Theorem 2.27. *Let $(X, \tau, \mathcal{I})$ be an ideal topological space. Then every subset of X is strongly g-$\star$-closed if and only if every $\star$-g-open set is closed.*

Proof. Suppose every subset of X is strongly g-$\star$-closed. Let U be any $\star$-g-open in X. Then $U \subseteq U$ and U is strongly g-$\star$-closed by assumption implies $\text{cl}(U) \subseteq U$. Hence U is closed.
Conversely, let $A \subseteq X$ and U be any $\star$-g-open such that $A \subseteq U$. Since U is closed by assumption, we have $\text{cl}(A) \subseteq \text{cl}(U) \subseteq U$. Thus A is strongly g-$\star$-closed. □

The following Theorem gives a characterization of normal spaces in terms of strongly g-$\star$-open sets.

Theorem 2.28. *Let $(X, \tau, \mathcal{I})$ be an ideal topological space. Then the following are equivalent.*

(1) X is normal,

(2) For any disjoint closed sets A and B, there exist disjoint strongly g-$\star$-open sets U and V such that $A \subseteq U$ and $B \subseteq V$,

(3) For any closed set A and open set V containing A, there exists a strongly g-$\star$-open set U such that $A \subseteq U \subseteq cl(U) \subseteq V$.

Proof.

(1)⇒(2) The proof follows from the fact that every open set is strongly g-$\star$-open.

(2)⇒(3) Suppose A is closed and V is an open set containing A. Since A and X−V are disjoint closed sets, there exist disjoint strongly g-$\star$-open sets U and W such that A⊆U and X−V⊆W. Since X−V is $\star$-g-closed and W is strongly g-$\star$-open, X−V⊆int(W). Then X−int(W)⊆V. Again U∩W=ϕ which implies that U∩int(W)=ϕ and so U⊆X−int(W). Then cl(U)⊆X−int(W)⊆V and thus U is the required strongly g-$\star$-open set with A⊆U⊆cl(U)⊆V.

(3)⇒(1) Let A and B be two disjoint closed subsets of X. Then A is a closed set and X−B an open set containing A. By hypothesis, there exists a strongly g-$\star$-open set U such that A⊆U⊆cl(U)⊆X−B. Since U is strongly g-$\star$-open and A is $\star$-g-closed we have A⊆int(U). Hence A⊆int(U)=G and B⊆X−cl(U)=H. G and H are the required disjoint open sets containing A and B respectively, which proves (1). □

Corollary 2.29. *Let (X, τ, $\mathcal{I}$) be an ideal topological space. If A is a strongly g-$\star$-closed subset of X, then A is $\mathcal{I}$-compact.*

Proof. The proof follows from the fact that every strongly g-$\star$-closed set is g-closed by Theorem 2.2 and hence $\mathcal{I}_g$-closed by Lemma 1.8. By Theorem 1.20, A is $\mathcal{I}$-compact. □

3. $\star\star$-g-$\mathcal{I}$-locally closed Sets

Definition 3.1. *A subset A of an ideal topological space (X, τ, $\mathcal{I}$) is called a $\star\star$-g-$\mathcal{I}$-locally closed set (briefly, $\star\star$-g-$\mathcal{I}$-LC) if A=U∩V where U is $\star$-g-open and V is closed.*

Definition 3.2 ([1]). *A subset A of an ideal topological space (X, τ, $\mathcal{I}$) is called a $\star$-g-$\mathcal{I}$-locally closed set (briefly, $\star$-g-$\mathcal{I}$-LC) if A=U∩V where U is $\star$-g-open and V is $\star$-closed.*

Proposition 3.3. *Let (X, τ, $\mathcal{I}$) be an ideal topological space and A a subset of X. Then the following hold.*

(1) If A is closed, then A is $\star\star$-g-$\mathcal{I}$-LC-set.

(2) If A is $\star$-g-open, then A is $\star\star$-g-$\mathcal{I}$-LC-set.

(3) If A is a $\star\star$-g-$\mathcal{I}$-LC-set, then A is a $\star$-g-$\mathcal{I}$-LC-set.

The converses of Proposition 3.3 need not be true as shown in the following Examples.

Example 3.4.

(1) In Example 2.6, $\star\star$-g-$\mathcal{I}$-LC-sets are ϕ, X, {b}, {c}, {d}, {a, c}, {b, d}, {c, d}, {a, b, c}, {a, c, d} and closed sets are ϕ, X, {b}, {b, d}, {a, b, c}. Clearly {c} is a $\star\star$-g-$\mathcal{I}$-LC-set but it is not closed.

(2) In Example 2.6, $\star$-g-open sets are ϕ, X, {c}, {d}, {a, c}, {c, d}, {a, c, d}. Clearly {b} is a $\star\star$-g-$\mathcal{I}$-LC-set but it is not $\star$-g-open.

Example 3.5. *In Example 2.6, $\star$-g-$\mathcal{I}$-LC-sets are ϕ, X, {a}, {b}, {c}, {d}, {a, b}, {a, c}, {a, d}, {b, d}, {c, d}, {a, b, c}, {a, b, d}, {a, c, d}. Clearly {a} is a $\star$-g-$\mathcal{I}$-LC-set but it is not a $\star\star$-g-$\mathcal{I}$-LC-set.*

Theorem 3.6. *Let (X, τ, $\mathcal{I}$) be an ideal topological space. If A is a $\star\star$-g-$\mathcal{I}$-LC-set and B is a closed set, then A∩B is a $\star\star$-g-$\mathcal{I}$-LC-set.*

Proof. Let B be closed, then A∩B=(U∩V)∩B=U∩(V∩B), where V∩B is closed. Hence A∩B is a $\star\star$-g-$\mathcal{I}$-LC-set. □

Theorem 3.7. *A subset of an ideal topological space $(X, \tau, \mathcal{I})$ is closed if and only if it is $\star\star$-g-$\mathcal{I}$-LC and strongly g-$\star$-closed.*

Proof. Necessity is trivial. We prove only sufficiency. Let A be $\star\star$-g-$\mathcal{I}$-LC-set and strongly g-$\star$-closed set. Since A is $\star\star$-g-$\mathcal{I}$-LC, A=U∩V, where U is $\star$-g-open and V is closed. So, we have A=U∩V⊆U. Since A is strongly g-$\star$-closed, cl(A) ⊆ U. Also since A = U∩V⊆V and V is closed, we have cl(A) ⊆ V. Consequently, cl(A) ⊆U∩V = A and hence A is closed. □

Remark 3.8. *The notions of $\star\star$-g-$\mathcal{I}$-LC-set and strongly g-$\star$-closed set are independent.*

Example 3.9. *In Example 2.6, clearly $\{c\}$ is a $\star\star$-g-$\mathcal{I}$-LC-set but not strongly g-$\star$-closed.*

Example 3.10. *In Example 2.6, clearly $\{b,\ c\}$ is strongly g-$\star$-closed but not a $\star\star$-g-$\mathcal{I}$-LC-set.*

4. Decomposition of Continuity

Definition 4.1. *A function $f : (X, \tau, \mathcal{I}) \rightarrow (Y, \sigma)$ is said to be $\star\star$-g-$\mathcal{I}$-LC-continuous (resp. strongly g-$\star$-continuous) if $f^{-1}(A)$ is $\star\star$-g-$\mathcal{I}$-LC-set (resp. strongly $\mathcal{I}_g$-$\star$-closed) in $(X, \tau, \mathcal{I})$ for every closed set A of (Y, σ).*

Theorem 4.2. *A function $f : (X, \tau, \mathcal{I}) \rightarrow (Y, \sigma)$ is continuous if and only if it is $\star\star$-g-$\mathcal{I}$-LC-continuous and strongly g-$\star$-continuous.*

Proof. It is an immediate consequence of Theorem 3.7. □

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