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On Contra $g\beta$-Continuous Functions

Research Article

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Abstract: In this paper, we introduce and investigate the notion of contra $g\beta$-continuous functions by utilizing $g\beta$-closed sets [33]. We obtain fundamental properties of contra $g\beta$-continuous functions and discuss the relationships between contra $g\beta$-continuity and other related functions.

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1. Introduction

In 1996, Dontchev [9] introduced a new class of functions called contra-continuous functions. He defined a function f : X → Y to be contra-continuous if the pre image of every open set of Y is closed in X. In 2007, Caldas et al. [4] introduced and investigated the notion of contra g-continuity. In 1968, Zaitsev [36] introduced the notion of π-open sets as a finite union of regular open sets. This notion received a proper attention and some research articles came to existence. Dontchev and Noiri [10] introduced and investigated π-continuity and πg-continuity. Ekici and Baker [11] studied further properties of πg-closed sets and continuities. In 2007, Ekici [12] introduced and studied some new forms of continuities. In [17], Kalantan introduced and investigated π-normality. The digital n-space is not a metric space, since it is not T_1. But recently Takigawa and Maki [34] showed that in the digital n-space every closed set is π-open. Recently, Ekici [13] introduced and studied contra πg-continuous functions. In 2010, Caldas et. al. [7] introduced and studied contra πgp-continuity.

In this paper, we present a new generalization of contra-continuity called contra $g\beta$-continuity. It turns out that the notion of contra $g\beta$-continuity is a weaker form of contra β-continuity and a stronger form of contra $\pi g\beta$-continuity [28].

2. Preliminaries

Throughout this paper, spaces (X, τ) and (Y, σ) (or simply X and Y) always mean topological spaces on which no separation axioms are assumed unless explicitly stated. Let A be a subset of a space X. The closure of A and the interior of A are denoted by cl(A) and int(A), respectively. A subset A of X is said to be regular open [31] (resp. regular closed [31]) if A

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= int(cl(A)) (resp. A = cl(int(A))). The finite union of regular open sets is said to be π-open [36]. The complement of a π-open set is said to be π-closed [36].

Definition 2.1. *A subset A of a space X is said to be*

(1) pre-closed [21] if cl(int(A)) $\subseteq$ A;

(2) α-open [23] if A $\subseteq$ int(cl(int(A)));

(3) semi-open [18] if A $\subseteq$ cl(int(A));

(4) β-open [1] if A $\subseteq$ cl(int(cl(A)));

(5) β-closed [1] if int(cl(int(A))) $\subseteq$ A;

(6) g-closed [19] if cl(A) $\subseteq$ U whenever A $\subseteq$ U and U is open in X;

(7) gp-closed [26] if pcl(A) $\subseteq$ U whenever A $\subseteq$ U and U is open in X;

(8) $g\beta$-closed [33] if βcl(A) $\subseteq$ U whenever A $\subseteq$ U and U is open in X;

(9) πgp-closed [27] if pcl(A) $\subseteq$ U whenever A $\subseteq$ U and U is π-open in X;

(10) $\pi g\beta$-closed [32] if βcl(A) $\subseteq$ U whenever A $\subseteq$ U and U is π-open in X.

The complements of the above closed sets are called their respective open sets. The complements of the above open sets are called their respective closed sets. The intersection of all pre-closed (resp. β-closed) sets containing A is called pre-closure (resp. β-closure) of A and is denoted by pcl(A) (resp. βcl(A)). The family of all $g\beta$-open (resp. $g\beta$-closed, closed) sets of X containing a point $x \in X$ is denoted by $G\beta O(X, x)$ (resp. $G\beta C(X, x)$, $C(X, x)$). The family of all $g\beta$-open (resp. $g\beta$-closed, closed, semi-open, β-open) sets of X is denoted by $G\beta O(X)$ (resp. $G\beta C(X)$, $C(X)$, $SO(X)$, $\beta O(X)$). Let A be a subset of a space (X, τ). The set $\bigcap \{U \in \tau : A \subseteq U\}$ is called the kernel of A [22] and is denoted by ker(A).

Lemma 2.2 ([16])**.** *The following properties hold for subsets U and V of a space (X, τ).*

(1) $x \in$ ker(U) if and only if $U \cap F \neq \emptyset$ for any closed set $F \in C(X, x)$;

(2) U $\subseteq$ ker(U) and U = ker(U) if U is open in X;

(3) If U $\subseteq$ V, then ker(U) $\subseteq$ ker(V).

3. Contra $g\beta$-continuous Functions

Definition 3.1. *Let A be a subset of a space (X, τ).*

(1) The set $\bigcap \{F : F \text{ is } g\beta\text{-closed in } X : A \subseteq F\}$ is called the $g\beta$-closure of A and is denoted by $g\beta$-cl(A).

(2) The set $\bigcup \{F : F \text{ is } g\beta\text{-open in } X : A \supseteq F\}$ is called the $g\beta$-interior of A and is denoted by $g\beta$-int(A).

Lemma 3.2. *Let A be a subset of a space (X, τ), then*

(1) $g\beta$-cl(X$-$A) = X$-g\beta$-int(A);

(2) $x \in g\beta$-cl(A) if and only if $A \cap U \neq \emptyset$ for each $U \in G\beta O(X, x)$;

(3) If A is $g\beta$-closed in X, then A = $g\beta$-cl(A).

Remark 3.3. *If A = $g\beta$-cl(A), then A need not be a $g\beta$-closed.*

Example 3.4. *Let $X = \{a, b, c\}$ and $\tau = \{\emptyset, X, \{a\}\}$. Take $A = \{a\}$. Clearly $g\beta$-cl(A) = A but A is not $g\beta$-closed.*

Definition 3.5. *A function $f : X \to Y$ is called contra $g\beta$-continuous if $f^{-1}(V)$ is $g\beta$-closed in X for every open set V of Y.*

Theorem 3.6. *The following are equivalent for a function $f : X \to Y$:*

(1) f is contra $g\beta$-continuous;

(2) The inverse image of every closed set of Y is $g\beta$-open in X;

(3) For each $x \in X$ and each closed set V in Y with $f(x) \in V$, there exists a $g\beta$-open set U in X such that $x \in U$ and $f(U) \subseteq V$;

(4) $f(g\beta\text{-}cl(A)) \subseteq ker(f(A))$ for every subset A of X;

(5) $g\beta\text{-}cl(f^{-1}(B)) \subseteq f^{-1}(ker(B))$ for every subset B of Y.

Proof.
(1) ⇒ (2): Let U be any closed set of Y. Since Y/U is open, then by (1), it follows that $f^{-1}(Y/U) = X/f^{-1}(U)$ is $g\beta$-closed. This shows that $f^{-1}(U)$ is $g\beta$-open in X.
(1) ⇒ (3): Let $x \in X$ and V be a closed set in Y with $f(x) \in V$. By (1), it follows that $f^{-1}(Y/V) = X/f^{-1}(V)$ is $g\beta$-closed and so $f^{-1}(V)$ is $g\beta$-open. Take $U = f^{-1}(V)$, we obtain that $x \in U$ and $f(U) \subseteq V$.
(3) ⇒ (2): Let V be a closed set in Y with $x \in f^{-1}(V)$. Since $f(x) \in V$, by (3) there exists a $g\beta$-open set U in X containing x such that $f(U) \subseteq V$. It follows that $x \in U \subseteq f^{-1}(V)$. Hence $f^{-1}(V)$ is $g\beta$-open.
(2) ⇒ (4): Let A be any subset of X. Let $y \notin ker(f(A))$. Then by Lemma 2.2, there exist a closed set F containing y such that $f(A) \cap F = \emptyset$. We have $A \cap f^{-1}(F) = \emptyset$ and since $f^{-1}(F)$ is $g\beta$-open then we have $g\beta\text{-cl}(A) \cap f^{-1}(F) = \emptyset$. Hence we obtain $f(g\beta\text{-cl}(A)) \cap F = \emptyset$ and $y \notin f(g\beta\text{-cl}(A))$. Thus $f(g\beta\text{-cl}(A)) \subseteq ker(f(A))$.
(4) ⇒ (5): Let B be any subset of Y. By (4), $f(g\beta\text{-cl}(f^{-1}(B))) \subseteq ker(B)$ and $g\beta\text{-cl}(f^{-1}(B)) \subseteq f^{-1}(ker(B))$.
(5) ⇒ (1): Let B be any open set of Y. By (5), $g\beta\text{-cl}(f^{-1}(B)) \subseteq f^{-1}(ker(B)) = f^{-1}(B)$ and $g\beta\text{-cl}(f^{-1}(B)) = f^{-1}(B)$. So we obtain that $f^{-1}(B)$ is $g\beta$-closed in X. □

Definition 3.7. *A function $f : X \to Y$ is said to be*

(1) completely continuous [2] if $f^{-1}(V)$ is regular open in X for every open set V of Y;

(2) contra-continuous [9] (resp. contra pre-continuous [15], contra β-continuous [5]) if $f^{-1}(V)$ is closed (resp. pre-closed, β-closed) in X for every open set V of Y;

(3) contra g-continuous [4] (resp. contra gp-continuous [7]) if $f^{-1}(V)$ is g-closed (resp. gp-closed) in X for every open set V of Y;

(4) contra π-continuous [7] (resp. contra πgp-continuous [7], contra $\pi g\beta$-continuous [28] if $f^{-1}(V)$ is π-closed (resp. πgp-closed, $\pi g\beta$-closed) in X for every open set V of Y.

For the functions defined above, we have the following implications:

$$\begin{array}{ccc} \textbf{\textit{contra } } \pi\textbf{\textit{-continuity}} & & \\ \downarrow & & \\ \textbf{\textit{contra-continuity}} & \longrightarrow & \textbf{\textit{contra pre-continuity}} \\ \downarrow & & \downarrow \\ \textbf{\textit{contra }} g\textbf{\textit{-continuity}} & & \textbf{\textit{contra }} \beta\textbf{\textit{-continuity}} \\ \downarrow & & \downarrow \\ \textbf{\textit{contra }} gp\textbf{\textit{-continuity}} & \longrightarrow & \textbf{\textit{contra }} g\beta\textbf{\textit{-continuity}} \\ \downarrow & & \downarrow \\ \textbf{\textit{contra }} \pi gp\textbf{\textit{-continuity}} & \longrightarrow & \textbf{\textit{contra }} \pi g\beta\textbf{\textit{-continuity}} \end{array}$$

Remark 3.8. *None of these implications is reversible as shown by the following Examples and the related papers [5, 7, 28].*

Example 3.9. *Let $X = \{a, b, c\}$, $\tau = \{\emptyset, X, \{a\}\}$ and $\sigma = \{\emptyset, X, \{a\}, \{a, b\}\}$. Then the identity function $f : (X, \tau) \to (X, \sigma)$ is contra $\pi g\beta$-continuous but not contra $g\beta$-continuous.*

Example 3.10. *Let $X = \{a, b, c\}$, $\tau = \{\emptyset, X, \{a\}\}$ and $\sigma = \{\emptyset, X, \{a, b\}\}$. Then the identity function $f : (X, \tau) \to (X, \sigma)$ is contra $g\beta$-continuous but not contra β-continuous.*

Definition 3.11. *A function $f : X \to Y$ is said to be*

(1) $g\beta$-semiopen if $f(U) \in SO(Y)$ for every $g\beta$-open set U of X;

(2) contra-I($g\beta$)-continuous if for each $x \in X$ and each $F \in C(Y, f(x))$, there exists $U \in G\beta O(X, x)$ such that $int(f(U)) \subseteq F$.

(3) $g\beta$-continuous [6] if $f^{-1}(F)$ is $g\beta$-closed in X for every closed set F of Y.

Theorem 3.12. *If a function $f : X \to Y$ is contra-I($g\beta$)-continuous and $g\beta$-semiopen, then f is contra $g\beta$-continuous.*

Proof. Suppose that x ∈ X and F ∈ C(Y, f(x)). Since f is contra-I($g\beta$)-continuous, there exists U ∈ $G\beta O$(X, x) such that int(f(U)) ⊆ F. By hypothesis f is $g\beta$-semiopen, therefore f(U) ∈ SO(Y) and f(U) ⊆ cl(int(f(U))) ⊆ F. This shows that f is contra $g\beta$-continuous. □

Lemma 3.13. *For a subset A of (X, τ), the following statements are equivalent.*

(1) A is open and $g\beta$-closed;

(2) A is regular open.

Proof.
(1) ⇒ (2): Since A is open and $g\beta$-closed, βcl(A) ⊆ A. It implies that int(cl(int(A))) ⊆ A. Also, since A is α-open, A ⊆ int(cl(int(A))). Hence int(A) = A = int(cl(int(A))). Thus A is regular open.
(2) ⇒ (1): Since A is regular open, A is open and int(A) = A = int(cl(int(A))). Since int(cl(int(A))) ⊆ A, A is β-closed and so $g\beta$-closed. □

Theorem 3.14. *For a function $f : X \to Y$, the following statements are equivalent.*

(1) f is contra $g\beta$-continuous and continuous;

(2) f is completely continuous.

Proof.
(1) ⇒ (2): Let U be an open set in Y. Since f is contra $g\beta$-continuous and continuous, $f^{-1}(U)$ is $g\beta$-closed and open, by Lemma 3.13, $f^{-1}(U)$ is regular open. Then f is completely continuous.
(2) ⇒ (1): Let U be an open set in Y. Since f is completely continuous, $f^{-1}(U)$ is regular open, by Lemma 3.13, $f^{-1}(U)$ is $g\beta$-closed and open. Then f is contra $g\beta$-continuous and continuous. □

Definition 3.15.

(1) A subset A of a topological space X is said to be Q-set [20] if int(cl(A)) = cl(int(A)).

(2) Let $f : X \to Y$ be a function. Then f is called Q-continuous [35] (resp. perfectly continuous [24]) if $f^{-1}(U)$ is Q-set (resp. clopen) in X for each open set U of Y.

Lemma 3.16. *For a subset A of X, the following statements are equivalent:*

(1) A is clopen,

(2) A is open, Q-set and $g\beta$-closed.

Proof.
(1) ⇒ (2): Since A is clopen, A is open and closed. Thus A is Q-set. Since A is closed, A is $g\beta$-closed.
(2) ⇒ (1): Since A is open and $g\beta$-closed, by Lemma 3.13, A is regular open. Since A is regular open and Q-set, A = int(cl(A)) = cl(int(A)). Hence A is regular closed and so closed. Since A is open and closed, A is clopen. □

Theorem 3.17. *For a function $f : X \to Y$, the following statements are equivalent.*

(1) f is perfectly continuous;

(2) f is continuous, Q-continuous and contra $g\beta$-continuous.

Proof. It is obtained from the above Lemma. □

Theorem 3.18. *If a function $f : X \to Y$ is contra $g\beta$-continuous and Y is regular, then f is $g\beta$-continuous.*

Proof. Let x be an arbitrary point of X and U be an open set of Y containing f(x). Since Y is regular, there exists an open set W in Y containing f(x) such that cl(W) ⊆ U. Since f is contra $g\beta$-continuous, there exists $V \in G\beta O(X, x)$ such that f(V) ⊆ cl(W). Then f(V) ⊆ cl(W) ⊆ U. Hence f is $g\beta$-continuous. □

Theorem 3.19. *Let $\{X_i : i \in \Omega\}$ be any family of topological spaces. If a function $f : X \to \prod X_i$ is contra $g\beta$-continuous, then $Pr_i \ o \ f : X \to X_i$ is contra $g\beta$-continuous for each $i \in \Omega$, where Pr_i is the projection of $\prod X_i$ onto X_i.*

Proof. For a fixed $i \in \Omega$, let V_i be any open set of X_i. Since Pr_i is continuous, $Pr_i^{-1}(V_i)$ is open in $\prod X_i$. Since f is contra $g\beta$-continuous, $f^{-1}(Pr_i^{-1}(V_i)) = (Pr_i \ o \ f)^{-1}(V_i)$ is $g\beta$-closed in X. Therefore, $Pr_i \ o$ f is contra $g\beta$-continuous for each $i \in \Omega$. □

Theorem 3.20. *Let $f : X \to Y$ and $g : Y \to Z$ be a function. Then the following hold:*

(1) If f is contra $g\beta$-continuous and g is continuous, then $g \ o \ f : X \to Z$ is contra $g\beta$-continuous;

(2) If f is $g\beta$-continuous and g is contra-continuous, then $g \ o \ f : X \to Z$ is contra $g\beta$-continuous;

(3) If f is contra $g\beta$-continuous and g is contra-continuous, then $g \ o \ f : X \to Z$ is $g\beta$-continuous.

Definition 3.21. *A space (X, τ) is called $g\beta$-$T_{1/2}$ if every $g\beta$-closed set is β-closed.*

Theorem 3.22. *Let $f : X \to Y$ be a function. Suppose that X is a $g\beta$-$T_{1/2}$ space. Then the following are equivalent.*

(1) f is contra $g\beta$-continuous;

(2) f is contra β-continuous.

Definition 3.23. *For a space (X, τ), ${}_g\tau^\beta = \{U \subseteq X : g\beta\text{-}cl(X\backslash U) = X\backslash U\}$.*

Theorem 3.24. *Let (X, τ) be a space. Then*

(1) Every $g\beta$-closed set is β-closed (i.e. (X, τ) is $g\beta$-$T_{1/2}$) if and only if ${}_g\tau^\beta = \beta O(X)$;

(2) Every $g\beta$-closed set is closed if and only if ${}_g\tau^\beta = \tau$.

Proof.
(1) Let A $\in {}_g\tau^\beta$. Then $g\beta$-cl(X\A) = X\A. By hypothesis, βcl(X\A) = $g\beta$-cl(X\A) = X\A and hence A $\in \beta$O(X). Conversely, let A be a $g\beta$-closed set. Then $g\beta$-cl(A) = A and hence X\A $\in {}_g\tau^\beta = \beta$O(X), i.e. A is β-closed.
(2) Similar to (1). □

Theorem 3.25. *If ${}_g\tau^\beta = \tau$ in X, then for a function $f : X \to Y$ the following are equivalent:*

(1) f is contra $g\beta$-continuous;

(2) f is contra gp-continuous;

(3) f is contra g-continuous;

(4) f is contra-continuous.

Theorem 3.26. *If ${}_g\tau^\beta = \tau$ in X, then for a function $f : X \to Y$ the following are equivalent:*

(1) f is contra $g\beta$-continuous;

(2) f is contra β-continuous;

(3) f is contra pre-continuous;

(4) f is contra-continuous.

4. Properties of Contra $g\beta$-continuous Functions

Definition 4.1. *A space X is said to be $g\beta$-T_1 if for each pair of distinct points x and y in X, there exist $g\beta$-open sets U and V containing x and y respectively, such that $y \notin U$ and $x \notin V$.*

Definition 4.2. *A space X is said to be $g\beta$-T_2 if for each pair of distinct points x and y in X, there exist $U \in G\beta O(X, x)$ and $V \in G\beta O(X, y)$ such that $U \cap V = \emptyset$.*

Theorem 4.3. *Let X be a topological space. Suppose that for each pair of distinct points x_1 and x_2 in X, there exists a function f of X into a Urysohn space Y such that $f(x_1) \neq f(x_2)$. Moreover, let f be contra $g\beta$-continuous at x_1 and x_2. Then X is $g\beta$-T_2.*

Proof. Let x_1 and x_2 be any distinct points in X. Then suppose that there exist an Urysohn space Y and a function f : X → Y such that $f(x_1) \neq f(x_2)$ and f is contra $g\beta$-continuous at x_1 and x_2. Let $w = f(x_1)$ and $z = f(x_2)$. Then $w \neq z$. Since Y is Urysohn, there exist open sets U and V containing w and z, respectively such that $cl(U) \cap cl(V) = \emptyset$. Since f is contra $g\beta$-continuous at x_1 and x_2, then there exist $g\beta$-open sets A and B containing x_1 and x_2, respectively such that $f(A) \subseteq cl(U)$ and $f(B) \subseteq cl(V)$. So we have $A \cap B = \emptyset$ since $cl(U) \cap cl(V) = \emptyset$. Hence, X is $g\beta$-T_2. □

Corollary 4.4. *If f is a contra $g\beta$-continuous injection of a topological space X into a Urysohn space Y, then X is $g\beta$-T_2.*

Proof. For each pair of distinct points x_1 and x_2 in X and f is contra $g\beta$-continuous function of X into a Urysohn space Y such that $f(x_1) \neq f(x_2)$ because f is injective. Hence by Theorem 4.3, X is $g\beta$-T_2. □

Definition 4.5. *A space (X, τ) is said to be $g\beta$-connected if X cannot be expressed as the disjoint union of two non-empty $g\beta$-open sets.*

Remark 4.6. *Every $g\beta$-connected space is connected.*

Theorem 4.7. *For a space X, the following are equivalent:*

(1) X is $g\beta$-connected;

(2) The only subsets of X which are both $g\beta$-open and $g\beta$-closed are the empty set $\emptyset$ and X;

(3) Each contra $g\beta$-continuous function of X into a discrete space Y with at least two points is a constant function.

Proof. (1) ⇒ (2): Suppose $S \subset X$ is a proper subset which is both $g\beta$-open and $g\beta$-closed. Then its complement $X - S$ is also $g\beta$-open and $g\beta$-closed. Then $X = S \cup (X - S)$, a disjoint union of two non-empty $g\beta$-open sets which contradicts the fact that X is $g\beta$-connected. Hence, $S = \emptyset$ or X.

(2) ⇒ (1): Suppose $X = A \cup B$ where $A \cap B = \emptyset$, $A \neq \emptyset$, $B \neq \emptyset$ and A and B are $g\beta$-open. Since $A = X - B$, A is $g\beta$-closed. But by assumption $A = \emptyset$ or X, which is a contradiction. Hence (1) holds.

(2) ⇒ (3): Let f : X → Y be contra $g\beta$-continuous function where Y is a discrete space with at least two points. Then $f^{-1}(\{y\})$ is $g\beta$-closed and $g\beta$-open for each $y \in Y$ and $X = \bigcup \{f^{-1}(y) : y \in Y\}$. By hypothesis, $f^{-1}(\{y\}) = \emptyset$ or X. If $f^{-1}(\{y\}) = \emptyset$ for all $y \in Y$, then f is not a function. Also there cannot exist more than one $y \in Y$ such that $f^{-1}(\{y\}) = X$. Hence there exists only one $y \in Y$ such that $f^{-1}(\{y\}) = X$ and $f^{-1}(\{y_1\}) = \emptyset$ where $y \neq y_1 \in Y$. This shows that f is a constant function.

(3) ⇒ (2): Let P be a non-empty set which is both $g\beta$-open and $g\beta$-closed in X. Suppose f : X → Y is a contra $g\beta$-continuous function defined by $f(P) = \{a\}$ and $f(X \backslash P) = \{b\}$ where $a \neq b$ and $a, b \in Y$. By hypothesis, f is constant. Therefore P = X. □

Definition 4.8. *A subset A of a space (X, τ) is said to be $g\beta$-clopen if A is both $g\beta$-open and $g\beta$-closed.*

Theorem 4.9. *If f is a contra $g\beta$-continuous function from a $g\beta$-connected space X onto any space Y, then Y is not a discrete space.*

Proof. Suppose that Y is discrete. Let A be a proper non-empty open and closed subset of Y. Then $f^{-1}(A)$ is a proper non-empty $g\beta$-clopen subset of X which is a contradiction to the fact that X is $g\beta$-connected. □

Theorem 4.10. *If f : X → Y is a contra $g\beta$-continuous surjection and X is $g\beta$-connected, then Y is connected.*

Proof. Suppose that Y is not a connected space. There exist non-empty disjoint open sets U_1 and U_2 such that $Y = U_1 \cup U_2$. Therefore U_1 and U_2 are clopen in Y. Since f is contra $g\beta$-continuous, $f^{-1}(U_1)$ and $f^{-1}(U_2)$ are $g\beta$-open in X. Moreover, $f^{-1}(U_1)$ and $f^{-1}(U_2)$ are non-empty disjoint and $X = f^{-1}(U_1) \cup f^{-1}(U_2)$. This shows that X is not $g\beta$-connected. This contradicts that Y is not connected assumed. Hence Y is connected. □

Definition 4.11. *The graph G(f) of a function $f : X \to Y$ is said to be contra $g\beta$-graph if for each $(x, y) \in (X \times Y)\backslash G(f)$, there exist a $g\beta$-open set U in X containing x and a closed set V in Y containing y such that $(U \times V) \cap G(f) = \emptyset$.*

Lemma 4.12. *A graph G(f) of a function $f : X \to Y$ is contra $g\beta$-graph in $X \times Y$ if and only if for each $(x, y) \in (X \times Y)\backslash G(f)$, there exist a $U \in G\beta O(X)$ containing x and $V \in C(Y)$ containing y such that $f(U) \cap V = \emptyset$.*

Theorem 4.13. *If $f : X \to Y$ is contra $g\beta$-continuous and Y is Urysohn, G(f) is contra $g\beta$-graph in $X \times Y$.*

Proof. Let $(x, y) \in (X \times Y)\backslash G(f)$. It follows that $f(x) \neq y$. Since Y is Urysohn, there exist open sets V and W such that $f(x) \in V$, $y \in W$ and $cl(V) \cap cl(W) = \emptyset$. Since f is contra $g\beta$-continuous, there exist a $U \in G\beta O(X, x)$ such that $f(U) \subseteq cl(V)$ and $f(U) \cap cl(W) = \emptyset$. Hence G(f) is contra $g\beta$-graph in $X \times Y$. □

Theorem 4.14. *Let $f : X \to Y$ be a function and $g : X \to X \times Y$ the graph function of f, defined by $g(x) = (x, f(x))$ for every $x \in X$. If g is contra $g\beta$-continuous, then f is contra $g\beta$-continuous.*

Proof. Let U be an open set in Y, then $X \times U$ is an open set in $X \times Y$. It follows that $f^{-1}(U) = g^{-1}(X \times U) \in G\beta C(X)$. Thus f is contra $g\beta$-continuous. □

Definition 4.15. *A space (X, τ) is said to be submaximal [3] if every dense subset of X is open in X and extremally disconnected [25] if the closure of every open set is open.*

Note that (X, τ) is submaximal and extremally disconnected if and only if every β-open set in X is open [14].

Theorem 4.16. *If A and B are $g\beta$-closed sets in submaximal and extremally disconnected space (X, τ), then $A\cup B$ is $g\beta$-closed.*

Proof. Let $A\cup B\subseteq U$ and U be open in (X, τ). Since $A, B \subseteq U$ and A and B are $g\beta$-closed, $\beta cl(A)\subseteq U$ and $\beta cl(B)\subseteq U$. Since (X, τ) is submaximal and extremally disconnected, $\beta cl(F) = cl(F)$ for any set $F\subseteq X$. Now $\beta cl(A\cup B) = \beta cl(A) \cup \beta cl(B)\subseteq U$. Hence $A\cup B$ is $g\beta$-closed. □

Lemma 4.17. *Let (X, τ) be a topological space. If $U, V \in G\beta O(X)$ and X is submaximal and extremally disconnected space, then $U \cap V \in G\beta O(X)$.*

Proof. Let $U, V \in G\beta O(X)$. We have $X\backslash U, X\backslash V \in G\beta C(X)$. By Theorem 4.16, $(X\backslash U) \cup (X\backslash V) = X\backslash(U\cap V) \in G\beta C(X)$. Thus, $U\cap V \in G\beta O(X)$. □

Theorem 4.18. *If $f : X \to Y$ and $g : X \to Y$ are contra $g\beta$-continuous, X is submaximal and extremally disconnected and Y is Urysohn, then $K = \{x \in X : f(x) = g(x)\}$ is $g\beta$-closed in X.*

Proof. Let $x \in X\backslash K$. Then $f(x) \neq g(x)$. Since Y is Urysohn, there exist open sets U and V such that $f(x) \in U$, $g(x) \in V$ and $cl(U) \cap cl(V) = \emptyset$. Since f and g are contra $g\beta$-continuous, $f^{-1}(cl(U)) \in G\beta O(X)$ and $g^{-1}(cl(V)) \in G\beta O(X)$. Let $A = f^{-1}(cl(U))$ and $B = g^{-1}(cl(V))$. Then A and B contains x. Set $C = A \cap B$. C is $g\beta$-open in X. Hence $f(C) \cap g(C) = \emptyset$ and $x \notin g\beta\text{-}cl(K)$. Thus K is $g\beta$-closed in X. □

Definition 4.19. *A subset A of a topological space X is said to be $g\beta$-dense in X if $g\beta\text{-}cl(A) = X$.*

Theorem 4.20. *Let $f : X \to Y$ and $g : X \to Y$ be contra $g\beta$-continuous. If Y is Urysohn and $f = g$ on a $g\beta$-dense set A $\subseteq$ X, then $f = g$ on X.*

Proof. Since f and g are contra $g\beta$-continuous and Y is Urysohn, by Theorem 4.18, K = {x ∈ X : f(x) =g(x)} is $g\beta$-closed in X. We have f = g on $g\beta$-dense set A ⊆ X. Since A ⊆ K and A is $g\beta$-dense set in X, then X = $g\beta$-cl(A) ⊆ $g\beta$-cl(K) = K. Hence, f = g on X. □

Definition 4.21. *A space X is said to be weakly Hausdroff [29] if each element of X is an intersection of regular closed sets.*

Theorem 4.22. *If $f : X \to Y$ is a contra $g\beta$-continuous injection and Y is weakly Hausdroff, then X is $g\beta$-T_1.*

Proof. Suppose that Y is weakly Hausdroff. For any distinct points x_1 and x_2 in X, there exist regular closed sets U and V in Y such that $f(x_1) \in U$, $f(x_2) \notin U$, $f(x_1) \notin V$ and $f(x_2) \in V$. Since f is contra $g\beta$-continuous, $f^{-1}(U)$ and $f^{-1}(V)$ are $g\beta$-open subsets of X such that $x_1 \in f^{-1}(U)$, $x_2 \notin f^{-1}(U)$, $x_1 \notin f^{-1}(V)$ and $x_2 \in f^{-1}(V)$. This shows that X is $g\beta$-T_1. □

Theorem 4.23. *Let $f : X \to Y$ have a contra $g\beta$-graph. If f is injective, then X is $g\beta$-T_1.*

Proof. Let x_1 and x_2 be any two distinct points of X. Then, we have $(x_1, f(x_2)) \in (X \times Y)\backslash G(f)$. Then, there exist a $g\beta$-open set U in X containing x_1 and $F \in C(Y, f(x_2))$ such that $f(U) \cap F = \emptyset$. Hence $U \cap f^{-1}(F) = \emptyset$. Therefore, we have $x_2 \notin U$. This implies that X is $g\beta$-T_1. □

Definition 4.24. *A topological space X is said to be Ultra Hausdroff [30] if for each pair of distinct points x and y in X, there exist clopen sets A and B containing x and y, respectively such that $A \cap B = \emptyset$.*

Theorem 4.25. *Let $f : X \to Y$ be a contra $g\beta$-continuous injection. If Y is an Ultra Hausdroff space, then X is $g\beta$-T_2.*

Proof. Let x_1 and x_2 be any distinct points in X, then $f(x_1) \neq f(x_2)$ and there exist clopen sets U and V containing $f(x_1)$ and $f(x_2)$ respectively, such that $U \cap V = \emptyset$. Since f is contra $g\beta$-continuous, then $f^{-1}(U) \in G\beta O(X)$ and $f^{-1}(V) \in G\beta O(X)$ such that $f^{-1}(U) \cap f^{-1}(V) = \emptyset$. Hence, X is $g\beta$-T_2. □

Definition 4.26. *A topological space X is said to be*

(1) $g\beta$-normal if each pair of non-empty disjoint closed sets can be separated by disjoint $g\beta$-open sets.

(2) Ultra normal [30] if for each pair of non-empty disjoint closed sets can be separated by disjoint clopen sets.

Theorem 4.27. *If $f : X \to Y$ is a contra $g\beta$-continuous, closed injection and Y is Ultra normal, then X is $g\beta$-normal.*

Proof. Let F_1 and F_2 be disjoint closed subsets of X. Since f is closed and injective, $f(F_1)$ and $f(F_2)$ are disjoint closed subsets of Y. Since Y is Ultra normal, $f(F_1)$ and $f(F_2)$ are separated by disjoint clopen sets V_1 and V_2, respectively. Hence $F_i \subseteq f^{-1}(V_i)$, $f^{-1}(V_i) \in G\beta O(X, x)$ for i = 1, 2 and $f^{-1}(V_1) \cap f^{-1}(V_2) = \emptyset$ and thus X is $g\beta$-normal. □

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