



Corrections on ω -open Sets and Decompositions of Continuity*

Research Article

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Abstract: In this paper, we introduce some generalized classes of τ_ω in topological spaces and investigate their properties. Also we study the relations between the new generalized classes of τ_ω and the other established generalized classes of τ_ω . With the help of all such classes, we obtain some new decompositions of continuity.

MSC: 54C05, 54C08, 54C10.

Keywords: Semi- ω^* -regular set, pre- ω^* -regular set, β - ω^* -regular set, weak ω - \mathcal{AB} -set.

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1. Introduction

In 1982, Hdeib [2] introduced ω -closed sets and ω -open sets in topological spaces. In 2009, Noiri et al [3] introduced some weaker forms of ω -open sets and obtained some decompositions of continuity. Quite Recently, Ravi et al [4] introduced the notion of semi- ω -open sets in topological spaces. Further they studied the weaker and stronger forms of such classes. The properties of such classes were also studied in detail. In this paper, further study is carried out using the established generalized classes of τ_ω . Apart from this, some new generalizations of ω -open sets are introduced and investigated. Also, using all the generalized subsets of τ_ω , some new decompositions of continuity in topological spaces are obtained.

2. Preliminaries

Throughout this paper, \mathbb{R} (resp. \mathbb{N} , \mathbb{Q} , \mathbb{Q}^* , \mathbb{Z}) denotes the set of all real numbers (resp. the set of all natural numbers, the set of all rational numbers, the set of all irrational numbers, the set of all integers). By a space (X, τ) , we always mean a topological space (X, τ) with no separation properties assumed. If $H \subset X$, $\text{cl}(H)$ and $\text{int}(H)$ will, respectively, denote the closure and interior of H in (X, τ) . τ_u denotes the usual topology on \mathbb{R} .

Definition 2.1 ([5]). Let H be a subset of a space (X, τ) . A point p in X is called a condensation point of H if for each open set U containing p , $U \cap H$ is uncountable.

* This paper is published in the Bulletin of the International Mathematical Virtual Institute, volume 6, issue 2, year 2016, pages 143-155. Examples 3.5(2), 3.15(2), 4.7(2), 4.20(2) and 4.37(2) of this paper were found to be incorrect in the published paper. So the incorrections are rightly corrected out in this paper for readers.

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Definition 2.2 ([2]). A subset H of a space (X, τ) is called ω -closed if it contains all its condensation points. The complement of an ω -closed set is called ω -open.

It is well known that a subset W of a space (X, τ) is ω -open if and only if for each $x \in W$, there exists $U \in \tau$ such that $x \in U$ and $U - W$ is countable. The family of all ω -open sets, denoted by τ_ω , is a topology on X , finer than τ . The interior and closure operator in (X, τ_ω) are denoted by int_ω and cl_ω respectively.

Definition 2.3 ([3]). A subset H of a space (X, τ) is called

- (1). α - ω -open if $H \subset int_\omega(cl(int_\omega(H)))$;
- (2). pre- ω -open if $H \subset int_\omega(cl(H))$;
- (3). β - ω -open if $H \subset cl(int_\omega(cl(H)))$;
- (4). b- ω -open if $H \subset int_\omega(cl(H)) \cup cl(int_\omega(H))$.

Definition 2.4 ([4]). A subset H of a space (X, τ) is called semi- ω -open if $H \subset cl(int_\omega(H))$.

Remark 2.5 ([3, 4]). The diagram holds for any subset of a space (X, τ) :

$$\begin{array}{ccccccc}
 \text{open} & \longrightarrow & \alpha\text{-}\omega\text{-open} & \longrightarrow & \text{pre-}\omega\text{-open} & & \\
 \downarrow & & \downarrow & & \downarrow & & \\
 \omega\text{-open} & \longrightarrow & \text{semi-}\omega\text{-open} & \longrightarrow & \text{b-}\omega\text{-open} & \longrightarrow & \beta\text{-}\omega\text{-open}
 \end{array}$$

In this diagram, none of the implications is reversible.

Definition 2.6 ([3]). A subset H of a space (X, τ) is called

- (1). an ω -t-set if $int(H) = int_\omega(cl(H))$;
- (2). an ω - \mathcal{B} -set if $H = U \cap V$, where $U \in \tau$ and V is an ω -t-set.

Definition 2.7 ([4]). A subset H of a space (X, τ) is called an ω^* -t-set if $int_\omega(cl(H)) = int_\omega(H)$.

Lemma 2.8 ([1]). If U is an open set, then $cl(U \cap H) = cl(U \cap cl(H))$ and hence $U \cap cl(H) \subset cl(U \cap H)$ for any subset H .

3. On New Subsets of τ_ω

Definition 3.1. A subset H of a space (X, τ) is called

- (1). an $\omega^\#$ -t-set if $int(H) = cl(int_\omega(H))$;
- (2). an $\omega^\#$ - \mathcal{B} -set if $H = U \cap V$, where $U \in \tau$ and V is an $\omega^\#$ -t-set.

Remark 3.2.

- (1). In \mathbb{R} with usual topology τ_u , a subset H with $int(H) = \phi$ is an $\omega^\#$ -t-set if and only if $int(H) = \phi = int_\omega(H)$.
- (2). In \mathbb{R} with usual topology τ_u , there is no proper subset H , with $int(H) \neq \phi$ which is an $\omega^\#$ -t-set. (or) The only subset in \mathbb{R} , with nonempty interior, which is an $\omega^\#$ -t-set is \mathbb{R} itself.

Proof.

(1). Proof is direct from the definition.

(2). If H is a proper $\omega^\#$ -t-set in \mathbb{R} with $int(H) \neq \phi$, then $int(H) = cl(int_\omega(H)) = cl(int(H))$. Thus $int(H)$ is a proper clopen subset in \mathbb{R} contradicting the connectedness of \mathbb{R} . Hence the result. \square

Example 3.3. In \mathbb{R} with usual topology τ_u ,

(1). $H = [0, 1] \cap \mathbb{Q}$ is an $\omega^\#$ -t-set by (1) of Remark 3.2 since $int(H) = \phi = int_\omega(H)$.

(2). $H = \mathbb{Q}^*$ is not an $\omega^\#$ -t-set by (1) of Remark 3.2 since $int(H) = \phi \neq int_\omega(H)$.

(3). $H = (0, 1]$ is not an $\omega^\#$ -t-set by (2) of Remark 3.2 since $int(H) \neq \phi$.

Remark 3.4. In a space (X, τ) ,

(1). Every open set is an $\omega^\#$ - \mathcal{B} -set.

(2). Every $\omega^\#$ -t-set is an $\omega^\#$ - \mathcal{B} -set.

Example 3.5. In \mathbb{R} with usual topology τ_u ,

(1). $H = [0, 1] \cap \mathbb{Q}$ is an $\omega^\#$ - \mathcal{B} -set by (2) of Remark 3.4 since H is an $\omega^\#$ -t-set by (1) of Example 3.3.

(2). $H = \mathbb{Q}^*$ is not an $\omega^\#$ - \mathcal{B} -set. If $H = U \cap V$ where U is open and V is an $\omega^\#$ -t-set, then $H \subseteq V$ and $cl(int_\omega(H)) \subseteq cl(int_\omega(V)) = int(V)$ by assumption V . Thus $\mathbb{R} \subseteq int(V) \subseteq V$ and so $V = \mathbb{R}$. Then $H = U \cap \mathbb{R} = U$ which implies that H is open. This is a contradiction since $H = \mathbb{Q}^*$ is not open. Thus $H = \mathbb{Q}^*$ is not an $\omega^\#$ - \mathcal{B} -set.

Remark 3.6. The converses of (1) and (2) in Remark 3.4 are not true as seen from the following Example.

Example 3.7. In \mathbb{R} with usual topology τ_u ,

(1). $H = [0, 1] \cap \mathbb{Q}$ is an $\omega^\#$ - \mathcal{B} -set by (1) of Example 3.5. But H is not open.

(2). $H = (0, 1)$ is an $\omega^\#$ - \mathcal{B} -set by (1) of Remark 3.4. But H is not an $\omega^\#$ -t-set by (2) of Remark 3.2.

Proposition 3.8. If A and B are $\omega^\#$ -t-sets of a space (X, τ) , then $A \cap B$ is an $\omega^\#$ -t-set.

Proof. Let A and B be $\omega^\#$ -t-sets. Then we have $int(A \cap B) \subset int_\omega(A \cap B) \subset cl(int_\omega(A \cap B)) = cl(int_\omega(A) \cap int_\omega(B)) \subset cl(int_\omega(A)) \cap cl(int_\omega(B)) = int(A) \cap int(B) = int(A \cap B)$. Thus $int(A \cap B) = cl(int_\omega(A \cap B))$ and hence $A \cap B$ is an $\omega^\#$ -t-set. \square

Theorem 3.9. For a subset H of a space (X, τ) , the following are equivalent:

(1). H is open;

(2). H is semi- ω -open and an $\omega^\#$ - \mathcal{B} -set.

Proof.

(1) \Rightarrow (2): (2) follows by Remark 2.5 and (1) of Remark 3.4.

(2) \Rightarrow (1): Given H is semi- ω -open and an $\omega^\#$ - \mathcal{B} -set. So $H = U \cap V$ where U is open and $int(V) = cl(int_\omega(V))$. Then $H \subset U = int(U)$. Also H is semi- ω -open implies $H \subset cl(int_\omega(H)) \subset cl(int_\omega(V)) = int(V)$ by assumption. Thus $H \subset int(U) \cap int(V) = int(U \cap V) = int(H)$ and hence H is open. \square

Remark 3.10. *The concepts of semi- ω -openness and being an $\omega^\#$ - \mathcal{B} -set are independent.*

Example 3.11. *In \mathbb{R} with usual topology τ_u ,*

- (1). $H = \mathbb{Q}^*$ is semi- ω -open but not an $\omega^\#$ - \mathcal{B} -set, by (2) of Example 3.5.
- (2). $H = [0, 1] \cap \mathbb{Q}$ is an $\omega^\#$ - \mathcal{B} -set by (1) of Example 3.5. But H is not semi- ω -open.

Definition 3.12. *A subset H of a space (X, τ) is called*

- (1). an $\omega^\#$ - t_α -set if $\text{int}(H) = \text{cl}(\text{int}_\omega(\text{cl}(H)))$;
- (2). an $\omega^\#$ - \mathcal{B}_α -set if $H = U \cap V$, where $U \in \tau$ and V is an $\omega^\#$ - t_α -set.

Example 3.13.

- (1). In \mathbb{R} with usual topology τ_u , \mathbb{Z} is an $\omega^\#$ - t_α -set since $\text{int}(\mathbb{Z}) = \phi$ and $\text{cl}(\text{int}_\omega(\text{cl}(\mathbb{Z}))) = \text{cl}(\text{int}_\omega(\mathbb{Z})) = \text{cl}(\phi) = \phi$.
- (2). In \mathbb{R} with usual topology τ_u , $H = (0, 1)$ is not an $\omega^\#$ - t_α -set since $\text{int}(H) = (0, 1)$, but $\text{cl}(\text{int}_\omega(\text{cl}(H))) = \text{cl}(\text{int}_\omega([0, 1])) = \text{cl}((0, 1)) = [0, 1]$.
- (3). In \mathbb{R} with usual topology τ_u , $H = \mathbb{Q}^*$ is not an $\omega^\#$ - t_α -set since $\text{int}(\mathbb{Q}^*) = \phi$, but $\text{cl}(\text{int}_\omega(\text{cl}(\mathbb{Q}^*))) = \text{cl}(\text{int}_\omega(\mathbb{R})) = \text{cl}(\mathbb{R}) = \mathbb{R}$.

Remark 3.14. *In a space (X, τ) ,*

- (1). Every open set is $\omega^\#$ - \mathcal{B}_α -set.
- (2). Every $\omega^\#$ - t_α -set is $\omega^\#$ - \mathcal{B}_α -set.

Example 3.15. (1). In \mathbb{R} with usual topology τ_u , \mathbb{Z} is an $\omega^\#$ - t_α -set by (1) of Example 3.13 and hence an $\omega^\#$ - \mathcal{B}_α -set by (2) of Remark 3.14.

- (2). In \mathbb{R} with usual topology τ_u , $H = \mathbb{Q}^*$ is not an $\omega^\#$ - \mathcal{B}_α -set. If $H = U \cap V$ where U is open and V is an $\omega^\#$ - t_α -set, then $H \subseteq V$ and $\text{cl}(\text{int}_\omega(\text{cl}(H))) \subseteq \text{cl}(\text{int}_\omega(\text{cl}(V))) = \text{int}(V)$ by assumption on V . Thus $\mathbb{R} \subseteq \text{int}(V) \subseteq V$ and so $V = \mathbb{R}$. Then $H = U \cap \mathbb{R} = U$ which implies that H is open. This is a contradiction since $H = \mathbb{Q}^*$ is not open. Thus $H = \mathbb{Q}^*$ is not an $\omega^\#$ - \mathcal{B}_α -set.

Remark 3.16. *The converses of (1) and (2) in Remark 3.14 are not true as seen from the following Example.*

Example 3.17. (1). In \mathbb{R} with the topology $\tau = \{\phi, \mathbb{R}, \mathbb{Q}^*\}$, $H = \mathbb{Q}$ is an $\omega^\#$ - \mathcal{B}_α -set by (2) of Remark 3.14 since H is an $\omega^\#$ - t_α -set. But H is not open.

- (2). In \mathbb{R} with usual topology τ_u , $H = (0, 1)$ is an $\omega^\#$ - \mathcal{B}_α -set by (1) of Remark 3.14 since H is open but not an $\omega^\#$ - t_α -set by (2) of Example 3.13.

Proposition 3.18. *If A and B are $\omega^\#$ - t_α -sets of a space (X, τ) , then $A \cap B$ is an $\omega^\#$ - t_α -set.*

Proof. Let A and B be $\omega^\#$ - t_α -sets. Then we have $\text{int}(A \cap B) \subset \text{int}_\omega(A \cap B) \subset \text{int}_\omega(\text{cl}(A \cap B)) \subset \text{cl}(\text{int}_\omega(\text{cl}(A \cap B))) \subset \text{cl}(\text{int}_\omega(\text{cl}(A))) \cap \text{cl}(\text{int}_\omega(\text{cl}(B))) = \text{int}(A) \cap \text{int}(B) = \text{int}(A \cap B)$. Then $\text{int}(A \cap B) = \text{cl}(\text{int}_\omega(\text{cl}(A \cap B)))$ and hence $A \cap B$ is an $\omega^\#$ - t_α -set. \square

The following Examples show that β - ω -openness and being an $\omega^\#$ - \mathcal{B}_α -set are independent.

Example 3.19.

- (1). In \mathbb{R} with usual topology τ_u , $H = \mathbb{Q}^*$ is not an $\omega^\#$ - \mathcal{B}_α -set, by (2) of Example 3.15. But it is β - ω -open.
- (2). In \mathbb{R} with the topology $\tau = \{\phi, \mathbb{R}, \mathbb{Q}^*\}$, $H = \mathbb{Q}$ is an $\omega^\#$ - \mathcal{B}_α -set by (1) of Example 3.17. But H is not β - ω -open.

Theorem 3.20. For a subset H of a space (X, τ) , the following are equivalent:

- (1). H is open;
- (2). H is β - ω -open and an $\omega^\#$ - \mathcal{B}_α -set.

Proof.

(1) \Rightarrow (2): follows by Remark 2.5 and (1) of Remark 3.14.

(2) \Rightarrow (1): Given H is β - ω -open and an $\omega^\#$ - \mathcal{B}_α -set. So $H = U \cap V$ where U is open and V is an $\omega^\#$ - t_α -set. Then $H \subset U = \text{int}(U)$. Also H is β - ω -open implies $H \subset \text{cl}(\text{int}_\omega(\text{cl}(H))) \subset \text{cl}(\text{int}_\omega(\text{cl}(V))) = \text{int}(V)$ since V is an $\omega^\#$ - t_α -set. Thus $H \subset \text{int}(U) \cap \text{int}(V) = \text{int}(U \cap V) = \text{int}(H)$ and hence H is open. □

Definition 3.21. A subset H of a space (X, τ) is called a strong ω - \mathcal{B} -set if $H = U \cap V$, where $U \in \tau$ and V is a ω - t -set and $\text{int}_\omega(\text{cl}(V)) = \text{cl}(\text{int}_\omega(V))$.

Example 3.22.

- (1). In \mathbb{R} with the topology $\tau = \{\phi, \mathbb{R}, \mathbb{Q}^*\}$, $H = \mathbb{Q}$ is a strong ω - \mathcal{B} -set. Since $\text{int}(H) = \text{int}_\omega(\text{cl}(H)) = \text{cl}(\text{int}_\omega(H)) = \phi$, H is a ω - t -set with $\text{int}_\omega(\text{cl}(H)) = \text{cl}(\text{int}_\omega(H))$ and $H = \mathbb{R} \cap H$, H is a strong ω - \mathcal{B} -set.
- (2). In \mathbb{R} with the topology $\tau = \{\phi, \mathbb{R}, \mathbb{N}, \mathbb{Q}^*, \mathbb{Q}^* \cup \mathbb{N}\}$, $H = \mathbb{Q}$ is not a strong ω - \mathcal{B} -set. If $H = U \cap V$ where U is open and V is an ω - t -set with $\text{int}_\omega(\text{cl}(H)) = \text{cl}(\text{int}_\omega(H))$ then we have $H \subset U$ and \mathbb{R} is the only open set containing H . Hence $U = \mathbb{R}$ and $H = \mathbb{R} \cap V = V$. But this is a contradiction since $\text{int}_\omega(\text{cl}(H)) = \text{int}_\omega(\mathbb{Q}) = \mathbb{N} \neq \mathbb{Q} = \text{cl}(\text{int}_\omega(H))$. Thus H is not a strong ω - \mathcal{B} -set.

Proposition 3.23. In a space (X, τ) ,

- (1). Every open set is a strong ω - \mathcal{B} -set.
- (2). Every ω - t -set H with $\text{int}_\omega(\text{cl}(H)) = \text{cl}(\text{int}_\omega(H))$ is a strong ω - \mathcal{B} -set.

Proof. Proof follows directly from the definition of a strong ω - \mathcal{B} -set. □

Remark 3.24. The converses of (1) and (2) in Proposition 3.23 are not true as seen from the following Example.

Example 3.25.

- (1). In \mathbb{R} with the topology $\tau = \{\phi, \mathbb{R}, \mathbb{Q}^*\}$, $H = \mathbb{Q}$ is a strong ω - \mathcal{B} -set by (1) of Example 3.22. But H is not open.
- (2). In \mathbb{R} with usual topology τ_u , $H = (0, 1)$ is open and hence by (1) of Proposition 3.23, H is a strong ω - \mathcal{B} -set. But $\text{int}_\omega(\text{cl}(H)) = (0, 1) \neq [0, 1] = \text{cl}(\text{int}_\omega(H))$ and hence H is not a ω - t -set with $\text{int}_\omega(\text{cl}(H)) = \text{cl}(\text{int}_\omega(H))$.

Proposition 3.26. In a space (X, τ) , every strong ω - \mathcal{B} -set is a ω - \mathcal{B} -set.

Proof. Proof follows from the fact that an ω - t -set H with $\text{int}_\omega(\text{cl}(H)) = \text{cl}(\text{int}_\omega(H))$ is an ω - t -set. □

Remark 3.27. The converse of Proposition 3.26 is not true as seen from the following Example.

Example 3.28. In \mathbb{R} with the topology $\tau = \{\phi, \mathbb{R}, \mathbb{N}, \mathbb{Q}^*, \mathbb{Q}^* \cup \mathbb{N}\}$, $H = \mathbb{Q}$ is an ω - \mathcal{B} -set since H is an ω - t -set. But H is not a strong ω - \mathcal{B} -set by (2) of Example 3.22.

Theorem 3.29. For a subset H of a space (X, τ) , the following are equivalent:

- (1). H is open;
- (2). H is b - ω -open and a strong ω - \mathcal{B} -set.

Proof.

(1) \Rightarrow (2): (2) follows by Remark 2.5 and Proposition 3.23.

(2) \Rightarrow (1): Given H is b - ω -open and a strong ω - \mathcal{B} -set. Since H is a strong ω - \mathcal{B} -set, $H = U \cap V$ where U is open and V is an ω - t -set with $\text{int}_\omega(\text{cl}(V)) = \text{cl}(\text{int}_\omega(V))$. Then $H \subset U = \text{int}(U)$. Also H is b - ω -open implies $H \subset \text{int}_\omega(\text{cl}(H)) \cup \text{cl}(\text{int}_\omega(H)) \subset \text{int}_\omega(\text{cl}(V)) \cup \text{cl}(\text{int}_\omega(V)) = \text{int}(V)$ by assumption. Thus $H \subset \text{int}(U) \cap \text{int}(V) = \text{int}(U \cap V) = \text{int}(H)$ and hence H is open. \square

Remark 3.30. The concepts of b - ω -openness and being a strong ω - \mathcal{B} -set are independent.

Example 3.31.

- (1). In \mathbb{R} with the topology $\tau = \{\phi, \mathbb{R}, \mathbb{Q}^*\}$, $H = \mathbb{Q}$ is a strong- ω - \mathcal{B} -set by (1) of Example 3.22. But H is not b - ω -open.
- (2). In \mathbb{R} with the topology $\tau = \{\phi, \mathbb{R}, \mathbb{N}, \mathbb{Q}^*, \mathbb{Q}^* \cup \mathbb{N}\}$, $H = \mathbb{Q}$ is not a strong- ω - \mathcal{B} -set by (2) of Example 3.22. But H is b - ω -open.

4. Semi- ω^* -regular, Pre- ω^* -regular and β - ω^* -regular Sets

Definition 4.1. A subset H of a space (X, τ) is called semi- ω^* -regular if H is semi- ω -open and an ω - t -set.

Example 4.2. In \mathbb{R} with usual topology τ_u ,

- (1). $H = (0, 1]$ is semi- ω -open as well as an ω - t -set and hence is semi- ω^* -regular.
- (2). $H = \mathbb{Q}$ is not semi- ω -open and hence not semi- ω^* -regular.

Remark 4.3. In a space (X, τ) ,

- (1). Every semi- ω^* -regular set is semi- ω -open.
- (2). Every semi- ω^* -regular set is an ω - t -set.

Remark 4.4. The converses of (1) and (2) in Remark 4.3 are not true as seen from the following Example.

Example 4.5.

- (1). In \mathbb{R} with usual topology τ_u , $H = \mathbb{Q}^*$ is semi- ω -open but H is not semi- ω^* -regular because it is not an ω - t -set.
- (2). In \mathbb{R} with the topology $\tau = \{\phi, \mathbb{R}, \mathbb{Q}^*\}$, $H = \mathbb{Q}$ is an ω - t -set. But H is not semi- ω^* -regular because it is not semi- ω -open.

Definition 4.6. A subset H of a space (X, τ) is called an ω - \mathcal{SB} -set if $H = U \cap V$ where $U \in \tau$ and V is a semi- ω^* -regular.

Example 4.7. In \mathbb{R} with usual topology τ_u ,

- (1). $H = (0, 1]$ is semi- ω^* -regular by (1) of Example 4.2 and hence an ω -SB-set for $H = \mathbb{R} \cap H$.
- (2). $H = \mathbb{Q}^*$ is not an ω -SB-set. If $H = U \cap V$ where U is open and V is semi- ω^* -regular, then $H \subseteq V$ and $\text{int}_\omega(\text{cl}(H)) \subseteq \text{int}_\omega(\text{cl}(V)) = \text{int}(V)$ by assumption on V . Thus $\mathbb{R} \subseteq \text{int}(V) \subseteq V$ and so $V = \mathbb{R}$. Then $H = U \cap \mathbb{R} = U$ which implies that H is open. This is a contradiction since $H = \mathbb{Q}^*$ is not open. Thus $H = \mathbb{Q}^*$ is not an ω -SB-set.

Proposition 4.8. In a space (X, τ) ,

- (1). Every open set is an ω -SB-set.
- (2). Every semi- ω^* -regular set is an ω -SB-set.

Proof. This is obvious from the definition of an ω -SB-set. □

Remark 4.9. The converses of (1) and (2) in Proposition 4.8 are not true as seen from the following Example.

Example 4.10.

- (1). In \mathbb{R} with usual topology τ_u , $H = (0, 1]$ is semi- ω^* -regular by (1) of Example 4.2 and hence is an ω -SB-set. But H is not open.
- (2). In \mathbb{R} with the topology $\tau = \{\phi, \mathbb{R}, \mathbb{Q}^*\}$, $H = \mathbb{Q}^*$ is open and hence an ω -SB-set. But H is not semi- ω^* -regular by (2) of Example 4.5.

Theorem 4.11. For a subset H of a space (X, τ) , the following are equivalent:

- (1). H is open;
- (2). H is pre- ω -open and an ω -SB-set.

Proof.

(1) \Rightarrow (2): This is obvious by Remark 2.5 and (1) of Proposition 4.8.

(2) \Rightarrow (1): Given H is pre- ω -open and an ω -SB-set. Since H is an ω -SB-set, $H = U \cap V$ where U is open and V is semi- ω^* -regular. Then we have $H \subset U = \text{int}(U)$. Also H is pre- ω -open implies $H \subset \text{int}_\omega(\text{cl}(H)) \subset \text{int}_\omega(\text{cl}(V)) = \text{int}(V)$ for V is an ω -t-set being semi- ω^* -regular. Thus $H \subset \text{int}(U) \cap \text{int}(V) = \text{int}(U \cap V) = \text{int}(H)$ and hence H is open. □

Remark 4.12. The notions of pre- ω -openness and being an ω -SB-set are independent.

Example 4.13. In \mathbb{R} with usual topology τ_u ,

- (1). $H = (0, 1]$ is an ω -SB-set by (1) of Example 4.7 but not pre- ω -open.
- (2). $H = \mathbb{Q}^*$ is pre- ω -open but not an ω -SB-set by (2) of Example 4.7.

Definition 4.14. A subset H of a space (X, τ) is called pre- ω^* -regular if H is pre- ω -open and an $\omega^\#$ -t-set.

Example 4.15. In \mathbb{R} with usual topology τ_u ,

- (1). $H = \mathbb{Q}$ is an $\omega^\#$ -t-set and also it is pre- ω -open. Hence it is pre- ω^* -regular.
- (2). $H = \{1\}$ is not pre- ω^* -regular because it is not pre- ω -open.

Proposition 4.16. In a space (X, τ) ,

(1). Every pre- ω^* -regular set is a pre- ω -open.

(2). Every pre- ω^* -regular set is an $\omega^\#$ -t-set.

Remark 4.17. The converses of (1) and (2) in Proposition 4.16 are not true as seen from the following Example.

Example 4.18. In \mathbb{R} with usual topology τ_u ,

(1). $H = (0, 1)$ is pre- ω -open. But H is not pre- ω^* -regular because it is not an $\omega^\#$ -t-set by (2) of Remark 3.2.

(2). $H = \{1\}$ is an $\omega^\#$ -t-set but not pre- ω^* -regular by (2) of Example 4.15.

Definition 4.19. A subset H of a space (X, τ) is called an ω -SC-set if $H = U \cap V$ where $U \in \tau$ and V is pre- ω^* -regular.

Example 4.20. In \mathbb{R} with usual topology τ_u ,

(1). $H = \mathbb{Q}$ is pre- ω^* -regular by (1) of Example 4.15 and hence is an ω -SC-set.

(2). $H = \mathbb{Q}^*$ is not an ω -SC-set. If $H = U \cap V$ where U is open and V is pre- ω^* -regular, then $H \subseteq V$ and $cl(int_\omega(H)) \subseteq cl(int_\omega(V)) = int(V)$ by assumption on V . Thus $\mathbb{R} \subseteq int(V) \subseteq V$ and so $V = \mathbb{R}$. Then $H = U \cap \mathbb{R} = U$ which implies that H is open. This is a contradiction since $H = \mathbb{Q}^*$ is not open. Thus $H = \mathbb{Q}^*$ is not an ω -SC-set.

Proposition 4.21. In a space (X, τ) ,

(1). Every open set is an ω -SC-set.

(2). Every pre- ω^* -regular set is an ω -SC-set.

Proof. Proof follows directly from the definition of an ω -SC-set. □

Remark 4.22. The converses of (1) and (2) in Proposition 4.21 are not true as seen from the following Example.

Example 4.23. In \mathbb{R} with usual topology τ_u ,

(1). $H = \mathbb{Q}$ is ω -SC-set by (1) of Example 4.20. But H is not open.

(2). $H = (0, 1)$ is open and hence ω -SC-set but not pre- ω^* -regular by (1) of Example 4.18.

Theorem 4.24. For a subset H of a space (X, τ) , the following are equivalent:

(1). H is open;

(2). H is semi- ω -open and an ω -SC-set.

Proof.

(1) \Rightarrow (2): (2) follows from Remark 2.5 and (1) of Proposition 4.21.

(2) \Rightarrow (1): Given H is semi- ω -open and an ω -SC-set. Since H is an ω -SC-set, $H = U \cap V$ where U is open and V is pre- ω^* -regular. Then $H \subset U = int(U)$ and $H \subset V$. Also H is semi- ω -open implies $H \subset cl(int_\omega(H)) \subset cl(int_\omega(V)) = int(V)$ since V is an $\omega^\#$ -t-set being pre- ω^* -regular. Thus $H \subset int(U) \cap int(V) = int(U \cap V) = int(H)$ and hence H is open. □

Remark 4.25. The notions of semi- ω -openness and being an ω -SC-set are independent.

Example 4.26. In \mathbb{R} with usual topology τ_u ,

(1). $H = \mathbb{Q}^*$ is semi- ω -open but not an ω -SC-set by (2) of Example 4.20.

(2). $H = \mathbb{Q}$ is ω -SC-set by (1) of Example 4.20 but not semi- ω -open.

Definition 4.27. A subset H of a space (X, τ) is called

(1). an ω - t_α -set [3] if $\text{int}(H) = \text{int}_\omega(\text{cl}(\text{int}_\omega(H)))$.

(2). β - ω^* -regular if H is β - ω -open and an ω - t_α -set.

Example 4.28.

(1). In \mathbb{R} with usual topology τ_u , $H = (0, 1]$ is β - ω -open as well as ω - t_α -set and hence is β - ω^* -regular.

(2). In \mathbb{R} with the topology $\tau = \{\phi, \mathbb{R}, \mathbb{Q}^*\}$, $H = \mathbb{Q}$ is not β - ω -open and hence is not β - ω^* -regular.

(3). In \mathbb{R} with the topology $\tau = \{\phi, \mathbb{R}, \mathbb{Q}^*\}$, $H = \mathbb{Q}^*$ is not ω - t_α -set and hence is not β - ω^* -regular.

Proposition 4.29. In a space (X, τ) , every ω - t -set is an ω - t_α -set.

Proof. Let H be an ω - t -set. The $\text{int}(H) = \text{int}_\omega(\text{cl}(H))$. Now $\text{int}_\omega(\text{cl}(\text{int}_\omega(H))) \subset \text{int}_\omega(\text{cl}(H)) = \text{int}(H)$. We have $\text{int}(H) \subset \text{int}_\omega(H) \subset \text{int}_\omega(\text{cl}(\text{int}_\omega(H)))$. Thus we obtain that $\text{int}(H) = \text{int}_\omega(\text{cl}(\text{int}_\omega(H)))$ and so H is an ω - t_α -set. \square

Example 4.30. In \mathbb{R} with usual topology τ_u , $H = \mathbb{Q}$ is an ω - t_α -set but not an ω - t -set.

Proposition 4.31. In a space (X, τ) , every $\omega^\#$ - t -set is an ω - t_α -set.

Proof. Let H be an $\omega^\#$ - t -set. Then $\text{int}(H) = \text{cl}(\text{int}_\omega(H))$. Now $\text{int}_\omega(\text{cl}(\text{int}_\omega(H))) \subset \text{cl}(\text{int}_\omega(H)) = \text{int}(H)$. Also, $\text{int}(H) \subset \text{int}_\omega(H) \subset \text{int}_\omega(\text{cl}(\text{int}_\omega(H)))$. Thus we obtain that $\text{int}(H) = \text{int}_\omega(\text{cl}(\text{int}_\omega(H)))$ and so H is an ω - t_α -set. \square

Example 4.32. In \mathbb{R} with usual topology τ_u , $H = (0, 1]$ is an ω - t_α -set but not an $\omega^\#$ - t -set.

Proposition 4.33. In a space (X, τ) ,

(1). Every semi- ω^* -regular set is β - ω^* -regular.

(2). Every pre- ω^* -regular set is β - ω^* -regular.

(3). Every β - ω^* -regular set is a β - ω -open.

(4). Every β - ω^* -regular set is an ω - t_α -set.

Proof.

(1). Proof follows from the fact that every semi- ω -open set is β - ω -open and every ω - t -set is an ω - t_α -set.

(2). Proof follows from the fact that every pre- ω -open set is β - ω -open and every $\omega^\#$ - t -set is an ω - t_α -set.

(3), (4). The proofs follow from their definitions. \square

Remark 4.34. The converses of (1), (2), (3) and (4) in Proposition 4.33 are not true as seen from the following Example.

Example 4.35. In \mathbb{R} with usual topology τ_u ,

(1). $H = \mathbb{Q}$ is an ω - t_α -set by Example 4.30 and also β - ω -open. Hence it is β - ω^* -regular. But H is not semi- ω -open and hence is not semi- ω^* -regular.

(2). $H = (0, 1]$ is β - ω^* -regular by (1) of Example 4.28. But H is not pre- ω -open and hence not pre- ω^* -regular.

(3). $H = \mathbb{Q}^*$ is β - ω -open. But H is not an ω - t_α -set and hence is not β - ω^* -regular.

(4). In \mathbb{R} with the topology $\tau = \{\phi, \mathbb{R}, \mathbb{Q}^*\}$, $H = \mathbb{Q}$ is an ω - t_α -set. But H is not β - ω -open and hence is not β - ω^* -regular.

Definition 4.36. A subset H of a space (X, τ) is called a weak ω - \mathcal{AB} -set if $H = U \cap V$, where $U \in \tau$ and V is β - ω^* -regular.

Example 4.37. In \mathbb{R} with usual topology τ_u ,

(1). $H = (0, 1]$ is β - ω^* -regular by (1) of Example 4.28 and hence a weak ω - \mathcal{AB} -set for $H = \mathbb{R} \cap H$.

(2). $H = \mathbb{Q}^*$ is not a weak ω - \mathcal{AB} -set. If $H = U \cap V$ where U is open and V is β - ω^* -regular, then $H \subseteq V$ and $\text{int}_\omega(\text{cl}(\text{int}_\omega(H))) \subseteq \text{int}_\omega(\text{cl}(\text{int}_\omega(V))) = \text{int}(V)$ since V is an ω - t_α -set by assumption. Thus $\mathbb{R} \subseteq \text{int}(V) \subseteq V$ and so $V = \mathbb{R}$. Then $H = U \cap \mathbb{R} = U$ which implies that H is open. This is a contradiction since $H = \mathbb{Q}^*$ is not open. Thus $H = \mathbb{Q}^*$ is not a weak ω - \mathcal{AB} -set.

Proposition 4.38. In a space (X, τ) , every weak ω - \mathcal{AB} -set is β - ω -open.

Proof. Let H be a weak ω - \mathcal{AB} -set. Then $H = U \cap V$, where U is open and V is β - ω^* -regular. Hence V is β - ω -open. So $H = U \cap V \subset U \cap \text{cl}(\text{int}_\omega(\text{cl}(V))) \subset \text{cl}(U \cap \text{int}_\omega(\text{cl}(V))) = \text{cl}(\text{int}_\omega(U) \cap \text{int}_\omega(\text{cl}(V))) = \text{cl}(\text{int}_\omega(U \cap \text{cl}(V))) \subseteq \text{cl}(\text{int}_\omega(\text{cl}(U \cap V))) = \text{cl}(\text{int}_\omega(\text{cl}(H)))$ by Lemma 2.8. Hence H is β - ω -open. \square

Remark 4.39. The converse of Proposition 4.38 is not true as seen from the following Example.

Example 4.40. In \mathbb{R} with usual topology τ_u , $H = \mathbb{Q}^*$ is β - ω -open. But H is not a weak ω - \mathcal{AB} -set by (2) of Example 4.37.

Proposition 4.41. In a space (X, τ) ,

(1). Every open set is a weak ω - \mathcal{AB} -set.

(2). Every β - ω^* -regular set is a weak ω - \mathcal{AB} -set.

Remark 4.42. The converses of (1) and (2) in Proposition 4.41 are not true as seen from the following Example.

Example 4.43.

(1). In \mathbb{R} with usual topology τ_u , $H = (0, 1]$ is a weak ω - \mathcal{AB} -set by (1) of Example 4.37. But H is not open.

(2). In \mathbb{R} with the topology $\tau = \{\phi, \mathbb{R}, \mathbb{Q}^*\}$, $H = \mathbb{Q}^*$ is open and hence a weak ω - \mathcal{AB} -set. But H is not β - ω^* -regular by (3) of Example 4.28.

Theorem 4.44. For a subset H of a space (X, τ) , the following are equivalent:

(1). H is β - ω^* -regular.

(2). H is an ω - t_α -set and a weak ω - \mathcal{AB} -set.

Proof.

(1) \Rightarrow (2): Proof follows directly since every β - ω^* -regular set is an ω - t_α -set by definition and a weak ω - \mathcal{AB} -set by (2) of Proposition 4.41.

(2) \Rightarrow (1): Let H be an ω - t_α -set and a weak ω - \mathcal{AB} -set. Since H is a weak ω - \mathcal{AB} -set, by Proposition 4.38 H is β - ω -open. Thus H is an ω - t_α -set as well as β - ω -open. Hence H is β - ω^* -regular. \square

Remark 4.45. The concepts of being an ω - t_α -set and being a weak ω - \mathcal{AB} -set are independent.

Example 4.46. In \mathbb{R} with the topology $\tau = \{\phi, \mathbb{R}, \mathbb{Q}^*\}$,

(1). $H = \mathbb{Q}$ is an ω - t_α -set by (4) of Example 4.35. If H is a weak ω - \mathcal{AB} -set, then \mathbb{R} is the only open set containing H and $H = U \cap V$ where U is open and V is β - ω^* -regular. Then $H \subset U$. So $U = \mathbb{R}$. Thus $H = \mathbb{R} \cap V = V$ where V is β - ω^* -regular which implies H is β - ω^* -regular. This is a contradiction since $H = \mathbb{Q}$ is not β - ω^* -regular by (4) of Example 4.35. This concludes that H is not a weak ω - \mathcal{AB} -set.

(2). $H = \mathbb{Q}^*$ is open and hence a weak ω - \mathcal{AB} -set. But $H = \mathbb{Q}^*$ is not an ω - t_α -set by (3) of Example 4.28.

Theorem 4.47. For a subset H of a space (X, τ) , the following are equivalent:

- (1). H is open;
- (2). H is α - ω -open and a weak ω - \mathcal{AB} -set.

Proof.

(1) \Rightarrow (2): Proof follows directly by Remark 2.5 and by (1) of Proposition 4.41.

(2) \Rightarrow (1): Given H is α - ω -open and a weak ω - \mathcal{AB} -set. Since H is a weak ω - \mathcal{AB} -set, $H = U \cap V$ where U is open and V is β - ω^* -regular. Then $H \subset U = \text{int}(U)$ and $H \subset V$. Also H is α - ω -open implies $H \subset \text{int}_\omega(\text{cl}(\text{int}_\omega(H))) \subset \text{int}_\omega(\text{cl}(\text{int}_\omega(V))) = \text{int}(V)$ for V is an ω - t_α -set being β - ω^* -regular. Thus $H \subset \text{int}(U) \cap \text{int}(V) = \text{int}(U \cap V) = \text{int}(H)$ and hence H is open. \square

Remark 4.48. The notions of α - ω -openness and being a weak ω - \mathcal{AB} -set are independent.

Example 4.49. In \mathbb{R} with usual topology τ_u ,

- (1). $H = \mathbb{Q}^*$ is α - ω -open. But H is not weak ω - \mathcal{AB} -set by (2) of Example 4.37.
- (2). $H = (0, 1]$ is a weak ω - \mathcal{AB} -set by (1) of Example 4.37. But H is not α - ω -open.

5. Decompositions of Continuity

Definition 5.1 ([3]). A function $f : X \rightarrow Y$ is said to be pre- ω -continuous (resp. α - ω -continuous) if $f^{-1}(V)$ is pre- ω -open (resp. α - ω -open) in X for each open set V in Y .

Definition 5.2. A function $f : X \rightarrow Y$ is said to be semi- ω -continuous (resp. b - ω -continuous, β - ω -continuous, $\omega^\#$ - \mathcal{B} -continuous, $\omega^\#$ - \mathcal{B}_α -continuous, ω - \mathcal{SB} -continuous, ω - \mathcal{SC} -continuous, strongly ω - \mathcal{B} -continuous, ω - t_α -continuous, β - ω^* -regular continuous, weakly ω - \mathcal{AB} -continuous) if $f^{-1}(V)$ is semi- ω -open (resp. b - ω -open, β - ω -open, an $\omega^\#$ - \mathcal{B} -set, an $\omega^\#$ - \mathcal{B}_α -set, an ω - \mathcal{SB} -set, an ω - \mathcal{SC} -set, a strong ω - \mathcal{B} -set, an ω - t_α -set, β - ω^* -regular, a weak ω - \mathcal{AB} -set) in X for each open set V in Y .

Theorem 5.3. For a function $f : X \rightarrow Y$, the following are equivalent:

- (1). f is continuous.
- (2). f is semi- ω -continuous and $\omega^\#$ - \mathcal{B} -continuous.

Proof. This is an immediate consequence of Theorem 3.9. \square

Theorem 5.4. For a function $f : X \rightarrow Y$, the following are equivalent:

- (1). f is continuous.
- (2). f is β - ω -continuous and $\omega^\#$ - \mathcal{B}_α -continuous.

Proof. This is an immediate consequence of Theorem 3.20. \square

Theorem 5.5. For a function $f : X \rightarrow Y$, the following are equivalent:

- (1). f is continuous.
- (2). f is b - ω -continuous and strongly ω - \mathcal{B} -continuous.

Proof. This is an immediate consequence of Theorem 3.29. □

Theorem 5.6. For a function $f : X \rightarrow Y$, the following are equivalent:

- (1). f is continuous.
- (2). f is pre- ω -continuous and ω - \mathcal{SB} -continuous.

Proof. This is an immediate consequence of Theorem 4.11. □

Theorem 5.7. For a function $f : X \rightarrow Y$, the following are equivalent:

- (1). f is continuous.
- (2). f is semi- ω -continuous and ω - \mathcal{SC} continuous.

Proof. This is an immediate consequence of Theorem 4.24. □

Theorem 5.8. For a function $f : X \rightarrow Y$, the following are equivalent.

- (1). f is β - ω^* -regular continuous.
- (2). f is ω - t_α -continuous and weakly ω - \mathcal{AB} -continuous.

Proof. This is an immediate consequence of Theorem 4.44. □

Theorem 5.9. For a function $f : X \rightarrow Y$, the following are equivalent:

- (1). f is continuous.
- (2). f is α - ω -continuous and weakly ω - \mathcal{AB} -continuous.

Proof. This is an immediate consequence of Theorem 4.47. □

References

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- [1] R.Engelking, *General topology*, Heldermann Veriag Berlin, 2nd edition, (1989).
 - [2] H.Z.Hdeib, ω -closed mappings, *Revista Colomb. De Matem.*, 16(1982), 65-78.
 - [3] T.Noiri, A.Al-Omari and M.S.M.Noorani, *Weak forms of ω -open sets and decompositions of continuity*, *Eur. J. Pure Appl. Math.*, 2(2009), 73-84.
 - [4] O.Ravi, S.Satheesh Kanna, I.Rajasekaran and M.Paranjothi, *New generalized classes of τ_ω* , *Eur. J. Pure Appl. Math.*, 9(2)(2016), 152-164.
 - [5] S.Willard, *General Topology*, Addison-Wesley, Reading, Mass, USA, (1970).