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Strongly $\mathcal{I}_g$-$\star$-closed Sets

Research Article

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Abstract: In [14], the notion of strongly $\mathcal{I}_g$-$\star$-closed sets is introduced in ideal topological spaces. Characterizations and properties of strongly $\mathcal{I}_g$-$\star$-closed sets and strongly $\mathcal{I}_g$-$\star$-open sets are given. A characterization of normal spaces is given in terms of strongly $\mathcal{I}_g$-$\star$-open sets. Also, it is established that a strongly $\mathcal{I}_g$-$\star$-closed subset of an $\mathcal{I}$-compact space is $\mathcal{I}$-compact.

MSC: 54A05, 54A10.

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1. Introduction and Preliminaries

By a space, we always mean a topological space (X, τ) with no separation properties assumed. If H⊆X, cl(H) and int(H) will, respectively, denote the closure and interior of H in (X, τ). A subset H of a space (X, τ) is an α-open [17] (resp. semi-open [10], preopen [13]) set if H⊆int(cl(int(H))) (resp. H⊆cl(int(H)), H⊆int(cl(H))). The family of all α-open sets in (X, τ), denoted by τ^α, is a topology on X finer than τ. The closure of H in (X, τ^α) is denoted by α-cl(H).

Definition 1.1. *A subset H of a space (X, τ) is said to be g-closed [11] if cl(H)⊆U whenever H⊆U and U is open in X.*

An ideal $\mathcal{I}$ on a space (X, τ) is a nonempty collection of subsets of X which satisfies (i) A∈$\mathcal{I}$ and B⊆A⇒B∈$\mathcal{I}$ and (ii) A∈$\mathcal{I}$ and B∈$\mathcal{I}$⇒A∪B∈$\mathcal{I}$. Given a space (X, τ) with an ideal $\mathcal{I}$ on X and if $\wp$(X) is the set of all subsets of X, a set operator $(.)^*$: $\wp$(X)→$\wp$(X), called a local function [9] of A with respect to τ and $\mathcal{I}$, is defined as follows: for A⊆X, $A^*(\mathcal{I},\tau)$={x∈X | U∩A∉$\mathcal{I}$ for every U∈τ(x)} where τ(x)={U∈τ | x∈U}. We will make use of the basic facts about the local functions [[7], Theorem 2.3] without mentioning it explicitly. A Kuratowski closure operator $cl^*(.)$ for a topology $\tau^*(\mathcal{I},\tau)$, called the $\star$-topology, finer than τ is defined by $cl^*(A)=A\cup A^*(\mathcal{I},\tau)$ [19]. When there is no chance for confusion, we will simply write A^* for $A^*(\mathcal{I},\tau)$ and τ^* for $\tau^*(\mathcal{I},\tau)$. $int^*(A)$ will denote the interior of A in (X, τ^*). If $\mathcal{I}$ is an ideal on X, then (X, τ, $\mathcal{I}$) is called an ideal topological space or an ideal space. $\mathcal{N}$ is the ideal of all nowhere dense subsets in (X, τ).

Definition 1.2. *A subset H of an ideal topological space (X, τ, $\mathcal{I}$) is called $\star$-closed [7] (resp. $\star$-dense in itself [5]) if H^*⊆H (resp. H⊆H^*). The complement of a $\star$-closed set is called $\star$-open.*

Definition 1.3. *A subset H of an ideal topological space (X, τ, $\mathcal{I}$) is called $\mathcal{I}_g$-closed [2] if H^*⊆U whenever H⊆U and U is open.*

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Definition 1.4. *A subset H of an ideal space $(X, \tau, \mathcal{I})$ is said to be*

(1). $\star$-g-closed [12] if $cl(H) \subseteq U$ whenever $H \subseteq U$ and U is $\star$-open,

(2). $\star$-g-open [12] if its complement is $\star$-g-closed,

(3). strongly g-$\star$-closed [1] if $cl(H) \subseteq U$ whenever $H \subseteq U$ and U is $\star$-g-open.

Recall that every open set is $\star$-g-open but not conversely.

Remark 1.5 ([1]). *For a subset of an ideal topological space, the following properties hold:*

(1). Every closed set is strongly g-$\star$-closed but not conversely.

(2). Every strongly g-$\star$-closed set is g-closed but not conversely.

Definition 1.6. *An ideal $\mathcal{I}$ is said to be*

(1). codense [3] or τ-boundary [16] if $\tau \cap \mathcal{I} = \{\phi\}$,

(2). completely codense [3] if $PO(X) \cap \mathcal{I} = \{\phi\}$, where PO(X) is the family of all preopen sets in (X, τ).

Lemma 1.7 ([3]). *Every completely codense ideal is codense but not conversely.*

Lemma 1.8 ([7]). *Let $(X, \tau, \mathcal{I})$ be an ideal topological space and A, B subsets of X. Then the following properties hold:*

(1). $A \subseteq B \Rightarrow A^* \subseteq B^*$,

(2). $A^* = cl(A^*) \subseteq cl(A)$,

(3). $(A^*)^* \subseteq A^*$,

(4). $(A \cup B)^* = A^* \cup B^*$,

(5). $(A \cap B)^* \subseteq A^* \cap B^*$.

Lemma 1.9 ([18]). *Let $(X, \tau, \mathcal{I})$ be an ideal topological space and $A \subseteq X$. If $A \subseteq A^*$, then $A^* = cl(A^*) = cl(A) = cl^*(A)$.*

Lemma 1.10 ([18]). *Let $(X, \tau, \mathcal{I})$ be an ideal topological space. Then $\mathcal{I}$ is codense if and only if $G \subseteq G^*$ for every semi-open set G in X.*

Lemma 1.11 ([18]). *Let $(X, \tau, \mathcal{I})$ be an ideal topological space. If $\mathcal{I}$ is completely codense, then $\tau^* \subseteq \tau^\alpha$.*

Definition 1.12 ([2]). *An ideal topological space $(X, \tau, \mathcal{I})$ is called $T_{\mathcal{I}}$ if every $\mathcal{I}_g$-closed subset of X is $\star$-closed in X.*

Lemma 1.13 ([15]). *If $(X, \tau, \mathcal{I})$ is a $T_{\mathcal{I}}$ ideal topological space and A is an $\mathcal{I}_g$-closed set, then A is a $\star$-closed set.*

Lemma 1.14 ([2]). *Every g-closed set is $\mathcal{I}_g$-closed but not conversely.*

2. Properties of Strongly $\mathcal{I}_g$-$\star$-closed Sets

Definition 2.1. *A subset A of an ideal topological space $(X, \tau, \mathcal{I})$ is said to be*

(1). strongly $\mathcal{I}_g$-$\star$-closed if $A^* \subseteq U$ whenever $A \subseteq U$ and U is $\star$-g-open,

(2). strongly $\mathcal{I}_g$-$\star$-open if its complement is strongly $\mathcal{I}_g$-$\star$-closed.

Theorem 2.2. *If $(X, \tau, \mathcal{I})$ is any ideal space, then every strongly $\mathcal{I}_g$-$\star$-closed set is $\mathcal{I}_g$-closed.*

Proof. It follows from the fact that every open set is $\star$-g-open. □

The converse of Theorem 2.2 is not true in general as shown in the following Example.

Example 2.3. *Let $X=\{a, b, c\}$, $\tau=\{\phi, X, \{a\}, \{b, c\}\}$ and $\mathcal{I}=\{\phi, \{a\}\}$. Then strongly $\mathcal{I}_g$-$\star$-closed sets are ϕ, X, $\{a\}$, $\{b, c\}$ and $\mathcal{I}_g$-closed sets are ϕ, X, $\{a\}$, $\{b\}$, $\{c\}$, $\{a, b\}$, $\{a, c\}$, $\{b, c\}$. Clearly $\{b\}$ is $\mathcal{I}_g$-closed but not strongly $\mathcal{I}_g$-$\star$-closed.*

Proposition 2.4. *If A is $\star$-g-closed of $(X, \tau, \mathcal{I})$ and B is closed in (X, τ), then $A \cap B$ is $\star$-g-closed in $(X, \tau, \mathcal{I})$.*

Proof. Let U be an $\star$-open in $(X, \tau, \mathcal{I})$ such that $A \cap B \subseteq U$. Put $W = X - B$. Then $A \subseteq U \cup W \in \tau^\star$. Since A is $\star$-g-closed, $cl(A) \subseteq U \cup W$. Now $cl(A \cap B) \subset cl(A) \cap cl(B) \subseteq (U \cup W) \cap B = (U \cap B) \cup (W \cap B) = U \cap B \subseteq U$. Hence $A \cap B$ is $\star$-g-closed. □

The following Theorem gives characterizations of strongly $\mathcal{I}_g$-$\star$-closed sets.

Theorem 2.5. *If $(X, \tau, \mathcal{I})$ is any ideal topological space and $A \subseteq X$, then the following are equivalent.*

(1). A is strongly $\mathcal{I}_g$-$\star$-closed,

(2). $cl^*(A) \subseteq U$ whenever $A \subseteq U$ and U is $\star$-g-open in X,

(3). $cl^*(A) - A$ contains no nonempty $\star$-g-closed set,

(4). $A^* - A$ contains no nonempty $\star$-g-closed set.

Proof. (1) ⇒ (2) Let $A \subseteq U$ where U is $\star$-g-open in X. Since A is strongly $\mathcal{I}_g$-$\star$-closed, $A^* \subseteq U$ and so $cl^*(A) = A \cup A^* \subseteq U$.

(2) ⇒ (3) Let F be a $\star$-g-closed subset such that $F \subseteq cl^*(A) - A$. Then $F \subseteq cl^*(A)$. Also $F \subseteq cl^*(A) - A \subseteq X - A$ and hence $A \subseteq X - F$ where $X - F$ is $\star$-g-open. By (2) $cl^*(A) \subseteq X - F$ and so $F \subseteq X - cl^*(A)$. Thus $F \subseteq cl^*(A) \cap X - cl^*(A) = \phi$.

(3) ⇒ (4) $A^* - A = A \cup A^* - A = cl^*(A) - A$ which has no nonempty $\star$-g-closed subset by (3).

(4) ⇒ (1) Let $A \subseteq U$ where U is $\star$-g-open. Then $X - U \subseteq X - A$ and so $A^* \cap (X - U) \subseteq A^* \cap (X - A) = A^* - A$. Since A^* is always a closed subset and $X - U$ is $\star$-g-closed, $A^* \cap (X - U)$ is a $\star$-g-closed set contained in $A^* - A$ and hence $A^* \cap (X - U) = \phi$ by (4). Thus $A^* \subseteq U$ and A is strongly $\mathcal{I}_g$-$\star$-closed. □

Theorem 2.6. *Every $\star$-closed set is strongly $\mathcal{I}_g$-$\star$-closed.*

Proof. Let A be a $\star$-closed. To prove A is strongly $\mathcal{I}_g$-$\star$-closed, let U be any $\star$-g-open set such that $A \subseteq U$. Since A is $\star$-closed, $A^* \subseteq A \subseteq U$. Thus A is strongly $\mathcal{I}_g$-$\star$-closed.

□

The converse of Theorem 2.6 is not true in general as shown in the following Example.

Example 2.7. *Let $X=\{a, b, c, d\}$, $\tau=\{\phi, X, \{d\}, \{a, c\}, \{a, c, d\}\}$ and $\mathcal{I}=\{\phi, \{a\}, \{d\}, \{a, d\}\}$. Then strongly $\mathcal{I}_g$-$\star$-closed sets are ϕ, X, $\{a\}$, $\{b\}$, $\{d\}$, $\{a, b\}$, $\{a, d\}$, $\{b, c\}$, $\{b, d\}$, $\{a, b, c\}$, $\{a, b, d\}$, $\{b, c, d\}$ and $\star$-closed sets are ϕ, X, $\{a\}$, $\{b\}$, $\{d\}$, $\{a, b\}$, $\{a, d\}$, $\{b, d\}$, $\{a, b, c\}$, $\{a, b, d\}$. Clearly $\{b, c\}$ is strongly $\mathcal{I}_g$-$\star$-closed but not $\star$-closed.*

Theorem 2.8. *Let $(X, \tau, \mathcal{I})$ be an ideal topological space. For every $A \in \mathcal{I}$, A is strongly $\mathcal{I}_g$-$\star$-closed.*

Proof. Let $A \in \mathcal{I}$ and let $A \subseteq U$ where U is $\star$-g-open. Since $A \in \mathcal{I}$, $A^* = \phi \subseteq U$. Thus A is strongly $\mathcal{I}_g$-$\star$-closed. □

Theorem 2.9. *If (X, τ, $\mathcal{I}$) is an ideal topological space, then A^* is always strongly $\mathcal{I}_g$-$\star$-closed for every subset A of X.*

Proof. Let $A^* \subseteq U$ where U is $\star$-g-open. Since $(A^*)^* \subseteq A^*$ [7], we have $(A^*)^* \subseteq U$. Hence A^* is strongly $\mathcal{I}_g$-$\star$-closed. □

Theorem 2.10. *Let (X, τ, $\mathcal{I}$) be an ideal topological space. Then every strongly $\mathcal{I}_g$-$\star$-closed, $\star$-g-open set is $\star$-closed.*

Proof. Let A be strongly $\mathcal{I}_g$-$\star$-closed and $\star$-g-open. We have $A \subseteq A$ where A is $\star$-g-open. Since A is strongly $\mathcal{I}_g$-$\star$-closed, $A^* \subseteq A$. Thus A is $\star$-closed. □

Corollary 2.11. *If (X, τ, $\mathcal{I}$) is a $T_{\mathcal{I}}$ ideal topological space and A is a strongly $\mathcal{I}_g$-$\star$-closed set, then A is $\star$-closed set.*

Proof. By assumption A is strongly $\mathcal{I}_g$-$\star$-closed in (X, τ, $\mathcal{I}$) and so by Theorem 2.2, A is $\mathcal{I}_g$-closed. Since (X, τ, $\mathcal{I}$) is a $T_{\mathcal{I}}$-space, by Definition 1.12, A is $\star$-closed. □

Corollary 2.12. *Let (X, τ, $\mathcal{I}$) be an ideal topological space and A be a strongly $\mathcal{I}_g$-$\star$-closed set. Then the following are equivalent.*

(1). A is a $\star$-closed set,

(2). $cl^*(A) - A$ is a $\star$-g-closed set,

(3). $A^* - A$ is a $\star$-g-closed set.

Proof. (1) ⇒ (2) By (1) A is $\star$-closed. Hence $A^* \subseteq A$ and $cl^*(A) - A = (A \cup A^*) - A = \phi$ which is a $\star$-g-closed set.
(2) ⇒ (3) $A^* - A = A \cup A^* - A = cl^*(A) - A$ which is a $\star$-g-closed set by (2).
(3) ⇒ (1) Since A is strongly $\mathcal{I}_g$-$\star$-closed, by Theorem 2.5(4), $A^* - A$ contains no non-empty $\star$-g-closed set. By assumption (3) $A^* - A$ is $\star$-g-closed and hence $A^* - A = \phi$. Thus $A^* \subseteq A$ and A is $\star$-closed. □

Theorem 2.13. *Let (X, τ, $\mathcal{I}$) be an ideal topological space. Then every strongly g-$\star$-closed set is a strongly $\mathcal{I}_g$-$\star$-closed set.*

Proof. Let A be a strongly g-$\star$-closed set. Let U be any $\star$-g-open set such that $A \subseteq U$. Since A is strongly g-$\star$-closed, $cl(A) \subseteq U$. So, $A^* \subseteq cl(A) \subseteq U$ and thus A is strongly $\mathcal{I}_g$-$\star$-closed. □

The converse of Theorem 2.13 is not true in general as shown in the following Example.

Example 2.14. *In Example 2.7, strongly g-$\star$-closed sets are ϕ, X, {b}, {a, b}, {b, c}, {b, d}, {a, b, c}, {a, b, d}, {b, c, d}. Clearly {a} is strongly $\mathcal{I}_g$-$\star$-closed but not strongly g-$\star$-closed.*

Theorem 2.15. *If (X, τ, $\mathcal{I}$) is an ideal topological space and A is a $\star$-dense in itself, strongly $\mathcal{I}_g$-$\star$-closed subset of X, then A is strongly g-$\star$-closed.*

Proof. Let $A \subseteq U$ where U is $\star$-g-open. Since A is strongly $\mathcal{I}_g$-$\star$-closed, $A^* \subseteq U$. As A is $\star$-dense in itself, by Lemma 1.9, $cl(A) = A^*$. Thus $cl(A) \subseteq U$ and hence A is strongly g-$\star$-closed. □

Corollary 2.16. *If (X, τ, $\mathcal{I}$) is any ideal topological space where $\mathcal{I}=\{\phi\}$, then A is strongly $\mathcal{I}_g$-$\star$-closed if and only if A is strongly g-$\star$-closed.*

Proof. In (X, τ, $\mathcal{I}$), if $\mathcal{I} = \{\phi\}$ then $A^* = cl(A)$ for the subset A. A is strongly $\mathcal{I}_g$-$\star$-closed ⇔ $A^* \subseteq U$ whenever $A \subseteq U$ and U is $\star$-g-open ⇔ $cl(A) \subseteq U$ whenever $A \subseteq U$ and U is $\star$-g-open ⇔ A is strongly g-$\star$-closed. □

Corollary 2.17. *In an ideal topological space (X, τ, $\mathcal{I}$) where $\mathcal{I}$ is codense, if A is a semi-open and strongly $\mathcal{I}_g$-$\star$-closed subset of X, then A is strongly g-$\star$-closed.*

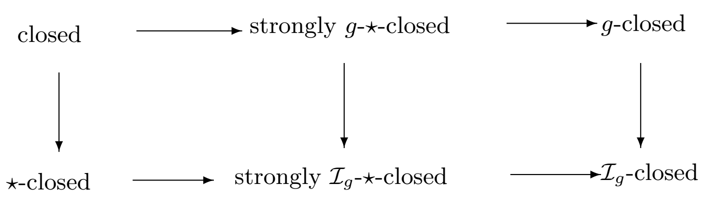
Proof. By Lemma 1.10, A is $\star$-dense in itself. By Theorem 2.15, A is strongly g-$\star$-closed. □

Example 2.18. *In Example 2.3, g-closed sets are ϕ, X, $\{a\}$, $\{b\}$, $\{c\}$, $\{a, b\}$, $\{a, c\}$, $\{b, c\}$. Clearly $\{b\}$ is g-closed but not strongly $\mathcal{I}_g$-$\star$-closed.*

Example 2.19. *In Example 2.7, g-closed sets are ϕ, X, $\{b\}$, $\{a, b\}$, $\{b, c\}$, $\{b, d\}$, $\{a, b, c\}$, $\{a, b, d\}$, $\{b, c, d\}$. Clearly $\{a\}$ is strongly $\mathcal{I}_g$-$\star$-closed but not g-closed.*

Remark 2.20. *We see that from Examples 2.18 and 2.19, g-closed sets and strongly $\mathcal{I}_g$-$\star$-closed sets are independent.*

Remark 2.21. *We have the following implications for the subsets stated above.*



Theorem 2.22. *Let $(X, \tau, \mathcal{I})$ be an ideal topological space and $A \subseteq X$. Then A is strongly $\mathcal{I}_g$-$\star$-closed if and only if $A = F - N$ where F is $\star$-closed and N contains no nonempty $\star$-g-closed set.*

Proof. If A is strongly $\mathcal{I}_g$-$\star$-closed, then by Theorem 2.5(4), $N = A^* - A$ contains no nonempty $\star$-g-closed set. If $F = cl^*(A)$, then F is $\star$-closed such that $F - N = (A \cup A^*) - (A^* - A) = (A \cup A^*) \cap (A^* \cap A^c)^c = (A \cup A^*) \cap ((A^*)^c \cup A) = (A \cup A^*) \cap (A \cup (A^*)^c) = A \cup (A^* \cap (A^*)^c) = A$.

Conversely, suppose $A = F - N$ where F is $\star$-closed and N contains no nonempty $\star$-g-closed set. Let U be an $\star$-g-open set such that $A \subseteq U$. Then $F - N \subseteq U$ which implies that $F \cap (X - U) \subseteq N$. Now $A \subseteq F$ and $F^* \subseteq F$ then $A^* \subseteq F^*$ and so $A^* \cap (X - U) \subseteq F^* \cap (X - U) \subseteq F \cap (X - U) \subseteq N$. Since $A^* \cap (X - U)$ is $\star$-g-closed, by hypothesis $A^* \cap (X - U) = \phi$ and so $A^* \subseteq U$. Hence A is strongly $\mathcal{I}_g$-$\star$-closed. □

Theorem 2.23. *Let $(X, \tau, \mathcal{I})$ be an ideal topological space and $A, B \subseteq X$. If $A \subseteq B \subseteq A^*$, then $A^* = B^*$ and B is $\star$-dense in itself.*

Proof. Since $A \subseteq B$, then $A^* \subseteq B^*$ and since $B \subseteq A^*$, then $B^* \subseteq (A^*)^* \subseteq A^*$. Therefore $A^* = B^*$ and $B \subseteq A^* \subseteq B^*$. Hence B is $\star$-dense in itself. □

Theorem 2.24. *Let $(X, \tau, \mathcal{I})$ be an ideal topological space. If A and B are subsets of X such that $A \subseteq B \subseteq cl^*(A)$ and A is strongly $\mathcal{I}_g$-$\star$-closed, then B is strongly $\mathcal{I}_g$-$\star$-closed.*

Proof. Since A is strongly $\mathcal{I}_g$-$\star$-closed, then by Theorem 2.5(3), $cl^*(A) - A$ contains no nonempty $\star$-g-closed set. But $cl^*(B) - B \subseteq cl^*(A) - A$ and so $cl^*(B) - B$ contains no nonempty $\star$-g-closed set. Hence B is strongly $\mathcal{I}_g$-$\star$-closed. □

Corollary 2.25. *Let $(X, \tau, \mathcal{I})$ be an ideal topological space. If A and B are subsets of X such that $A \subseteq B \subseteq A^*$ and A is strongly $\mathcal{I}_g$-$\star$-closed, then A and B are strongly g-$\star$-closed sets.*

Proof. Let A and B be subsets of X such that $A \subseteq B \subseteq A^*$. Then $A \subseteq B \subseteq A^* \subseteq cl^*(A)$. Since A is strongly $\mathcal{I}_g$-$\star$-closed, by Theorem 2.24, B is strongly $\mathcal{I}_g$-$\star$-closed. Since $A \subseteq B \subseteq A^*$, we have $A^* = B^*$. Hence $A \subseteq A^*$ and $B \subseteq B^*$. Thus A is $\star$-dense in itself and B is $\star$-dense in itself and by Theorem 2.15, A and B are strongly g-$\star$-closed. □

The following Theorem gives a characterization of strongly $\mathcal{I}_g$-$\star$-open sets.

Theorem 2.26. *Let (X, τ, $\mathcal{I}$) be an ideal topological space and A$\subseteq$X. Then A is strongly $\mathcal{I}_g$-$\star$-open if and only if F$\subseteq$int*(A) whenever F is $\star$-g-closed and F$\subseteq$A.*

Proof. Suppose A is strongly $\mathcal{I}_g$-$\star$-open. If F is $\star$-g-closed and F$\subseteq$A, then X$-$A$\subseteq$X$-$F and so cl*(X$-$A)$\subseteq$X$-$F by Theorem 2.5(2). Therefore F$\subseteq$X$-$cl*(X$-$A)=int*(A). Hence F$\subseteq$int*(A).

Conversely, suppose the condition holds. Let U be an $\star$-g-open set such that X$-$A$\subseteq$U. Then X$-$U$\subseteq$A and so X$-$U$\subseteq$int*(A). Therefore cl*(X$-$A)$\subseteq$U. By Theorem 2.5(2), X$-$A is strongly $\mathcal{I}_g$-$\star$-closed. Hence A is strongly $\mathcal{I}_g$-$\star$-open. □

Corollary 2.27. *Let (X, τ, $\mathcal{I}$) be an ideal topological space and A$\subseteq$X. If A is strongly $\mathcal{I}_g$-$\star$-open, then F$\subseteq$int*(A) whenever F is closed and F$\subseteq$A.*

The following Theorem gives a property of strongly $\mathcal{I}_g$-$\star$-closed.

Theorem 2.28. *Let (X, τ, $\mathcal{I}$) be an ideal topological space and A,B$\subseteq$X. If A is strongly $\mathcal{I}_g$-$\star$-open and int*(A)$\subseteq$B$\subseteq$A, then B is strongly $\mathcal{I}_g$-$\star$-open.*

Proof. Since int*(A) $\subseteq$ B $\subseteq$ A, we have X $-$ A $\subseteq$ X $-$ B $\subseteq$ X $-$ int*(A) = cl*(X $-$ A). By assumption A is strongly $\mathcal{I}_g$-$\star$-open and so X $-$ A is strongly $\mathcal{I}_g$-$\star$-closed. Hence by Theorem 2.24, X $-$ B is strongly $\mathcal{I}_g$-$\star$-closed and B is strongly $\mathcal{I}_g$-$\star$-open. □

The following Theorem gives a characterization of strongly $\mathcal{I}_g$-$\star$-closed sets in terms of strongly $\mathcal{I}_g$-$\star$-open sets.

Theorem 2.29. *Let (X, τ, $\mathcal{I}$) be an ideal topological space and A$\subseteq$X. Then the following are equivalent.*

(1). A is strongly $\mathcal{I}_g$-$\star$-closed,

(2). A$\cup$(X$-$A*) is strongly $\mathcal{I}_g$-$\star$-closed,

(3). A$^*-$A is strongly $\mathcal{I}_g$-$\star$-open.

Proof. (1) $\Rightarrow$ (2). Suppose A is strongly $\mathcal{I}_g$-$\star$-closed. If U is any $\star$-g-open set such that (A$\cup$(X$-$A*))$\subseteq$U, then X$-$U$\subseteq$X$-$(A$\cup$(X$-$A*))=[A$\cup$(A*)^c]^c=A$^*\cap$A^c=A$^*-$A. Since A is strongly $\mathcal{I}_g$-$\star$-closed, by Theorem 2.5(4), it follows that X$-$U=ϕ and so X=U. Since X is the only $\star$-g-open set containing A$\cup$(X$-$A*), clearly, A$\cup$(X$-$A*) is strongly $\mathcal{I}_g$-$\star$-closed.
(2) $\Rightarrow$ (1). Suppose A$\cup$(X$-$A*) is strongly $\mathcal{I}_g$-$\star$-closed. If F is any $\star$-g-closed set such that F$\subseteq$A$^*-$A = X$-$(A$\cup$(X$-$A*)), then A$\cup$(X$-$A*)$\subseteq$X$-$F and X$-$F is $\star$-g-open. Therefore, (A$\cup$(X$-$A*))$^*\subseteq$X$-$F which implies that A$^*\cup$(X$-$A*)$^*\subseteq$X$-$F and so F$\subseteq$X$-$A*. Since F$\subseteq$A*, it follows that F=ϕ. Hence A is strongly $\mathcal{I}_g$-$\star$-closed.
The equivalence of (2) and (3) follows from the fact that X$-$(A$^*-$A)=A$\cup$(X$-$A*). □

Theorem 2.30. *Let (X, τ, $\mathcal{I}$) be an ideal topological space. Then every subset of X is strongly $\mathcal{I}_g$-$\star$-closed if and only if every $\star$-g-open set is $\star$-closed.*

Proof. Suppose every subset of X is strongly $\mathcal{I}_g$-$\star$-closed. Let U be any $\star$-g-open in X. Then U $\subseteq$ U and U is strongly $\mathcal{I}_g$-$\star$-closed by assumption implies U* $\subseteq$ U. Hence U is $\star$-closed.
Conversely, let A $\subseteq$ X and U be any $\star$-g-open such that A $\subseteq$ U. Since U is $\star$-closed by assumption, we have A* $\subseteq$ U* $\subseteq$ U. Thus A is strongly $\mathcal{I}_g$-$\star$-closed. □

The following Theorem gives a characterization of normal spaces in terms of strongly $\mathcal{I}_g$-$\star$-open sets.

Theorem 2.31. *Let (X, τ, $\mathcal{I}$) be an ideal topological space where $\mathcal{I}$ is completely codense. Then the following are equivalent.*

(1). X is normal,

(2). For any disjoint closed sets A and B, there exist disjoint strongly $\mathcal{I}_g$-$\star$-open sets U and V such that A$\subseteq$U and B$\subseteq$V,

(3). For any closed set A and open set V containing A, there exists a strongly $\mathcal{I}_g$-$\star$-open set U such that A$\subseteq$U$\subseteq$cl(U)$\subseteq$V.*

Proof. (1)⇒(2) The proof follows from the fact that every open set is strongly $\mathcal{I}_g$-$\star$-open.
(2)⇒(3) Suppose A is closed and V is an open set containing A. Since A and X−V are disjoint closed sets, there exist disjoint strongly $\mathcal{I}_g$-$\star$-open sets U and W such that A$\subseteq$U and X−V$\subseteq$W. Since X−V is $\star$-g-closed and W is strongly $\mathcal{I}_g$-$\star$-open, X−V$\subseteq$int*(W). Then X−int*(W)$\subseteq$V. Again U∩W=ϕ which implies that U∩int*(W)=ϕ and so U$\subseteq$X−int*(W). Then cl*(U)$\subseteq$X−int*(W)$\subseteq$V and thus U is the required strongly $\mathcal{I}_g$-$\star$-open set with A$\subseteq$U$\subseteq$cl*(U)$\subseteq$V.
(3)⇒(1) Let A and B be two disjoint closed subsets of X. Then A is a closed set and X − B an open set containing A. By hypothesis, there exists a strongly $\mathcal{I}_g$-$\star$-open set U such that A$\subseteq$U$\subseteq$cl*(U)$\subseteq$X−B. Since U is strongly $\mathcal{I}_g$-$\star$-open and A is $\star$-g-closed we have A$\subseteq$int*(U). Since $\mathcal{I}$ is completely codense, by Lemma 1.11, $\tau^*\subseteq\tau^\alpha$ and so int*(U) and X−cl*(U)$\in\tau^\alpha$. Hence A$\subseteq$int*(U)$\subseteq$ int(cl(int(int*(U))))=G and B$\subseteq$X−cl*(U)$\subseteq$ int(cl(int(X−cl*(U)))) =H. G and H are the required disjoint open sets containing A and B respectively, which proves (1). □

Definition 2.32. *A subset H of an ideal topological space $(X, \tau, \mathcal{I})$ is said to be an $\star$-$g\alpha$-closed set if α-cl(H)$\subseteq$U whenever H$\subseteq$U and U is $\star$-g-open.*
The complement of an $\star$-$g\alpha$-closed set is said to be an $\star$-$g\alpha$-open set.

If $\mathcal{I}$=$\mathcal{N}$, it is not difficult to see that strongly $\mathcal{I}_g$-$\star$-closed sets coincide with $\star$-$g\alpha$-closed sets and so we have the following Corollary.

Corollary 2.33. *Let (X, τ, $\mathcal{I}$) be an ideal topological space where $\mathcal{I}$=$\mathcal{N}$. Then the following are equivalent.*

(1). X is normal,

(2). For any disjoint closed sets A and B, there exist disjoint $\star$-$g\alpha$-open sets U and V such that A$\subseteq$U and B$\subseteq$V,

(3). For any closed set A and open set V containing A, there exists an $\star$-$g\alpha$-open set U such that A$\subseteq$U$\subseteq$$cl_\alpha$(U)$\subseteq$V.

Definition 2.34. *A subset H of an ideal topological space is said to be $\mathcal{I}$-compact [4] or compact modulo $\mathcal{I}$ [16] if for every open cover $\{U_\alpha \mid \alpha\in\Delta\}$ of H, there exists a finite subset Δ_0 of Δ such that $H-\cup\{U_\alpha \mid \alpha\in\Delta_0\}\in\mathcal{I}$. The space (X, τ, $\mathcal{I}$) is $\mathcal{I}$-compact if X is $\mathcal{I}$-compact as a subset.*

Theorem 2.35 ([15])**.** *Let (X, τ, $\mathcal{I}$) be an ideal topological space. If A is an $\mathcal{I}_g$-closed subset of X, then A is $\mathcal{I}$-compact.*

Corollary 2.36. *Let (X, τ, $\mathcal{I}$) be an ideal topological space. If A is a strongly $\mathcal{I}_g$-$\star$-closed subset of X, then A is $\mathcal{I}$-compact.*

Proof. The proof follows from the fact that every strongly $\mathcal{I}_g$-$\star$-closed is $\mathcal{I}_g$-closed. □

3. $\star$-g-$\mathcal{I}$-locally Closed Sets

Definition 3.1. *A subset A of an ideal topological space (X, τ, $\mathcal{I}$) is called a $\star$-g-$\mathcal{I}$-locally closed set (briefly, $\star$-g-$\mathcal{I}$-LC) if A=U∩V where U is $\star$-g-open and V is $\star$-closed.*

Definition 3.2 ([8]). *A subset A of an ideal topological space (X, τ, $\mathcal{I}$) is called a weakly $\mathcal{I}$-locally closed set (briefly, weakly $\mathcal{I}$-LC) if A=U∩V where U is open and V is $\star$-closed.*

Proposition 3.3. *Let (X, τ, $\mathcal{I}$) be an ideal topological space and A a subset of X. Then the following hold.*

(1). If A is $\star$-g-open, then A is $\star$-g-$\mathcal{I}$-LC-set.

(2). If A is $\star$-closed, then A is $\star$-g-$\mathcal{I}$-LC-set.

(3). If A is a weakly $\mathcal{I}$-LC-set, then A is a $\star$-g-$\mathcal{I}$-LC-set.

The converses of Proposition 3.3 need not be true as shown in the following Examples.

Example 3.4.

(1). In Example 2.7, $\star$-g-$\mathcal{I}$-LC-sets are ϕ, X, $\{a\}$, $\{b\}$, $\{c\}$, $\{d\}$, $\{a, b\}$, $\{a, c\}$, $\{a, d\}$, $\{b, d\}$, $\{c, d\}$, $\{a, b, c\}$, $\{a, b, d\}$, $\{a, c, d\}$ and $\star$-closed sets are ϕ, X, $\{a\}$, $\{b\}$, $\{d\}$, $\{a, b\}$, $\{a, d\}$, $\{b, d\}$, $\{a, b, c\}$, $\{a, b, d\}$. Clearly $\{c\}$ is a $\star$-g-$\mathcal{I}$-LC-set but it is not a $\star$-closed set.

(2). In Example 2.7, $\star$-g-open sets are ϕ, X, $\{c\}$, $\{d\}$, $\{a, c\}$, $\{c, d\}$, $\{a, c, d\}$. Clearly $\{a, b\}$ is a $\star$-g-$\mathcal{I}$-LC-set but it is not a $\star$-g-open set.

Example 3.5. *In Example 2.7, weakly $\mathcal{I}$-LC-sets are ϕ, X, $\{a\}$, $\{b\}$, $\{c\}$, $\{d\}$, $\{a, b\}$, $\{a, c\}$, $\{a, d\}$, $\{b, d\}$, $\{a, b, c\}$, $\{a, b, d\}$, $\{a, c, d\}$. It is clear that $\{c, d\}$ is a $\star$-g-$\mathcal{I}$-LC-set but it is not a weakly $\mathcal{I}$-LC-set.*

Theorem 3.6. *Let (X, τ, $\mathcal{I}$) be an ideal topological space. If A is a $\star$-g-$\mathcal{I}$-LC-set and B is a $\star$-closed set, then A∩B is a $\star$-g-$\mathcal{I}$-LC-set.*

Proof. Let B be $\star$-closed, then A∩B=(U∩V)∩B=U∩(V∩B), where V∩B is $\star$-closed. Hence A∩B is a $\star$-*g*-$\mathcal{I}$-LC-set. □

Theorem 3.7. *A subset of an ideal topological space (X, τ, $\mathcal{I}$) is $\star$-closed if and only if it is*

(i). weakly $\mathcal{I}$-LC and $\mathcal{I}_g$-closed [6].

(ii). $\star$-g-$\mathcal{I}$-LC and strongly $\mathcal{I}_g$-$\star$-closed.

Proof. (ii) Necessity is trivial. We prove only sufficiency. Let A be $\star$-*g*-$\mathcal{I}$-LC-set and strongly $\mathcal{I}_g$-$\star$-closed set. Since A is $\star$-*g*-$\mathcal{I}$-LC, A=U∩V, where U is $\star$-*g*-open and V is $\star$-closed. So, we have A=U∩V⊆U. Since A is strongly $\mathcal{I}_g$-$\star$-closed, $A^* \subseteq$ U. Also since A = U∩V⊆V and V is $\star$-closed, we have $A^* \subseteq$ V. Consequently, $A^* \subseteq$U∩V = A and hence A is $\star$-closed. □

Remark 3.8.

(1). The notions of weakly $\mathcal{I}$-LC-set and $\mathcal{I}_g$-closed set are independent [6].

(2). The notions of $\star$-g-$\mathcal{I}$-LC-set and strongly $\mathcal{I}_g$-$\star$-closed set are independent.

Example 3.9. *In Example 2.7, clearly $\{c\}$ is a $\star$-g-$\mathcal{I}$-LC-set but not strongly $\mathcal{I}_g$-$\star$-closed.*

Example 3.10. *In Example 2.7, clearly $\{b, c\}$ is strongly $\mathcal{I}_g$-$\star$-closed but not a $\star$-g-$\mathcal{I}$-LC-set.*

4. Decompositions of $\star$-continuity

Definition 4.1. *A function $f : (X, \tau, \mathcal{I}) \to (Y, \sigma)$ is said to be $\star$-continuous [6] (resp. $\mathcal{I}_g$-continuous [6], $\star$-g-$\mathcal{I}$-LC-continuous, strongly $\mathcal{I}_g$-$\star$-continuous, weakly $\mathcal{I}$-LC-continuous [8]) if $f^{-1}(A)$ is $\star$-closed (resp. $\mathcal{I}_g$-closed, $\star$-g-$\mathcal{I}$-LC-set, strongly $\mathcal{I}_g$-$\star$-closed, weakly $\mathcal{I}$-LC-set) in $(X, \tau, \mathcal{I})$ for every closed set A of (Y, σ).*

Theorem 4.2. *A function $f : (X, \tau, \mathcal{I}) \to (Y, \sigma)$ is $\star$-continuous if and only if it is (i) weakly $\mathcal{I}$-LC-continuous and $\mathcal{I}_g$-continuous [6]. (ii) $\star$-g-$\mathcal{I}$-LC-continuous and strongly $\mathcal{I}_g$-$\star$-continuous.*

Proof. It is an immediate consequence of Theorem 3.7. □

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