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Mildly $\mathcal{I}_g$-$\star$-closed Sets

Research Article

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Abstract: In this paper, another generalized class of τ^* called mildly $\mathcal{I}_g$-$\star$-open sets is introduced and the notion of mildly $\mathcal{I}_g$-$\star$-closed sets in ideal topological spaces is studied. The relationships of mildly $\mathcal{I}_g$-$\star$-closed sets with various like sets are investigated.

MSC: 54A05, 54A10.

Keywords: τ^*, generalized class, mildly $\mathcal{I}_g$-$\star$-closed set, ideal topological space, generalized closed set, $\mathcal{I}_g$-closed set, $pre^*_{\mathcal{I}}$-closed set, $pre^*_{\mathcal{I}}$-open set.

1. Introduction

The first step of generalizing closed sets was done by Levine in 1970 [7]. He defined a subset H of a topological space (X, τ) to be g-closed if its closure belongs to every open superset of H. As the weak form of g-closed sets, the notion of weakly g-closed sets was introduced and studied by Sundaram and Nagaveni [12]. Sundaram and Pushpalatha [13] introduced and studied the notion of strongly g-closed sets, which is implied by that of closed sets and implies that of g-closed sets. Park and Park [10] introduced and studied mildly g-closed sets, which is properly placed between the classes of strongly g-closed and weakly g-closed sets. Moreover, the relations with other notions directly or indirectly connected with g-closed were investigated by them. In 1999, Dontchev et al. studied the notion of generalized closed sets in ideal topological spaces called $\mathcal{I}_g$-closed sets [2]. In 2008, Navaneethakrishnan and Paulraj Joseph have studied some characterizations of normal spaces via $\mathcal{I}_g$-open sets [9]. In 2013, Ekici and Ozen [4] introduced a generalized class of τ^* in ideal topological spaces. In 2015, Mandal and Mukherjee [8] introduced and studied the notions of $\star$-g-closed and $\star$-g-open sets in ideal topological spaces. The main aim of this paper is to introduce another generalized class of τ^* called mildly $\mathcal{I}_g$-$\star$-open sets in ideal topological spaces and to study the notion of mildly $\mathcal{I}_g$-$\star$-closed sets in ideal topological spaces. Moreover, this generalized class of τ^* generalize $\mathcal{I}_g$-$\star$-open sets and mildly $\mathcal{I}_g$-$\star$-open sets. The relationships of mildly $\mathcal{I}_g$-$\star$-closed sets with various like sets are discussed.

2. Preliminaries

In this paper, (X, τ) represents topological space on which no separation axioms are assumed unless explicitly stated. The closure and the interior of a subset G of a topological space (X, τ) will be denoted by cl(G) and int(G), respectively. An

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ideal $\mathcal{I}$ on a topological space (X, τ) is a nonempty collection of subsets of X which satisfies

(1). M $\in \mathcal{I}$ and N $\subseteq$ M imply N $\in \mathcal{I}$ and

(2). M $\in \mathcal{I}$ and N $\in \mathcal{I}$ imply M$\cup$N $\in \mathcal{I}$ [6].

Given a topological space (X, τ) with an ideal $\mathcal{I}$ on X if $\mathcal{P}$(X) is the set of all subsets of X, a set operator $(\bullet)^\star : \mathcal{P}(X) \rightarrow \mathcal{P}(X)$, called a local function [6] of G with respect to τ and $\mathcal{I}$ is defined as follows: for G $\subseteq$ X, $G^\star(\mathcal{I}, \tau) = \{x \in X \mid U \cap G \notin \mathcal{I}$ for every $U \in \tau(x)\}$ where $\tau(x) = \{U \in \tau \mid x \in U\}$. A Kuratowski closure operator $cl^\star(\bullet)$ for a topology $\tau^\star(\mathcal{I}, \tau)$, called the $\star$-topology and finer than τ, is defined by $cl^\star(G) = G \cup G^\star(\mathcal{I}, \tau)$ [14]. We will simply write $G^\star$ for $G^\star(\mathcal{I}, \tau)$ and $\tau^\star$ for $\tau^\star(\mathcal{I}, \tau)$. If $\mathcal{I}$ is an ideal on X, then (X, τ, $\mathcal{I}$) is called an ideal topological space. On the other hand, (A, τ_A, $\mathcal{I}_A$) where τ_A is the relative topology on A and $\mathcal{I}_A = \{A \cap J : J \in \mathcal{I}\}$ is an ideal topological space for an ideal topological space (X, τ, $\mathcal{I}$) and A $\subseteq$ X [5]. For a subset G $\subseteq$ X, cl*(G) and int*(G) will, respectively, denote the closure and the interior of G in (X, τ*).

Definition 2.1 ([8])**.** *A subset G of an ideal topological space $(X, \tau, \mathcal{I})$ is said to be*

(1). $\star$-g-closed if cl(G) $\subseteq$ H whenever G $\subseteq$ H and H is $\star$-open in X;

(2). $\star$-g-open if X$\backslash$G is $\star$-g-closed.

Definition 2.2. *A subset G of an ideal topological space (X, τ, $\mathcal{I}$) is said to be*

(1). $\mathcal{I}_g$-closed [2, 9] if G $\subseteq$ H whenever G $\subseteq$ H and H is open in (X, τ, $\mathcal{I}$).*

(2). pre$^*_{\mathcal{I}}$-open [3] if G $\subseteq$ int(cl(G)).*

(3). pre$^*_{\mathcal{I}}$-closed [3] if X$\backslash$G is pre$^*_{\mathcal{I}}$-open (or) cl*(int(G)) $\subseteq$ G.

(4). $\mathcal{I}$-R closed [1] if G = cl(int(G)).*

(5). $\star$-closed [5] if G = cl(G) or G* $\subseteq$ G.*

Remark 2.3 ([4])**.** *In any ideal topological space, every $\mathcal{I}$-R closed set is $\star$-closed but not conversely.*

Definition 2.4 ([8])**.** *An ideal topological space $(X, \tau, \mathcal{I})$ is called $\star$-g-normal if for each pair of disjoint $\star$-g-closed subsets F and K of X, there exist disjoint open subsets U and V of X such that F $\subseteq$ U and K $\subseteq$ V.*

Remark 2.5 ([8])**.** *In any ideal topological space, every closed set is $\star$-g-closed but not conversely.*

3. Another Generalized Classes of τ*

Definition 3.1. *Let (X, τ, $\mathcal{I}$) be an ideal topological space. A subset G of (X, τ, $\mathcal{I}$) is said to be*

(1). mildly $\mathcal{I}_g$-$\star$-closed if (int(G)) $\subseteq$ H whenever G $\subseteq$ H and H is a $\star$-g-open set in X;*

(2). weakly $\mathcal{I}_g$-closed [11] if (int(G)) $\subseteq$ H whenever G $\subseteq$ H and H is an open set in X;*

(3). strongly $\mathcal{I}_g$-$\star$-closed if G $\subseteq$ H whenever G $\subseteq$ H and H is a $\star$-g-open set in X.*

Example 3.2. *Let (X, τ, $\mathcal{I}$) be an ideal topological space such that X = {a, b, c}, τ = {$\emptyset$, {a}, X} and $\mathcal{I}$ = {$\emptyset$}. Then {b} is a mildly $\mathcal{I}_g$-$\star$-closed set but {a} is not a mildly $\mathcal{I}_g$-$\star$-closed set.*

Definition 3.3. *Let $(X, \tau, \mathcal{I})$ be an ideal topological space and $G \subseteq X$. Then G is said to be a mildly $\mathcal{I}_g$-$\star$-open set (resp. a strongly $\mathcal{I}_g$-$\star$-open set, a weakly $\mathcal{I}_g$-open set) if $X \backslash G$ is a mildly $\mathcal{I}_g$-$\star$-closed set (resp. a strongly $\mathcal{I}_g$-$\star$-closed set, a weakly $\mathcal{I}_g$-closed set).*

Remark 3.4. *Let $(X, \tau, \mathcal{I})$ be an ideal topological space. The following diagram holds for a subset $G \subseteq X$:*

$$\begin{array}{ccc} \text{strongly } \mathcal{I}_g\text{-}\star\text{-closed set} & \longrightarrow & \mathcal{I}_g\text{-closed set} \\ \downarrow & & \downarrow \\ \text{mildly } \mathcal{I}_g\text{-}\star\text{-closed set} & \longrightarrow & \text{weakly } \mathcal{I}_g\text{-closed set} \end{array}$$

These implications are not reversible as shown in the following Examples.

Example 3.5. *(a) Let X, τ and $\mathcal{I}$ be as in Example 3.2. Then*

(1) $\{a, b\}$ is a weakly $\mathcal{I}_g$-closed set but not mildly $\mathcal{I}_g$-$\star$-closed;

(2) $\{b\}$ is a mildly $\mathcal{I}_g$-$\star$-closed set but not strongly $\mathcal{I}_g$-$\star$-closed;

(3) $\{b\}$ is an $\mathcal{I}_g$-closed set but not strongly $\mathcal{I}_g$-$\star$-closed.

(b) Let $(X, \tau, \mathcal{I})$ be an ideal topological space such that $X = \{a, b, c, d\}$, $\tau = \{\emptyset, \{a\}, \{a, b\}, \{a, c\}, \{a, b, c\}, X\}$ and $\mathcal{I} = \{\emptyset, \{a\}, \{b\}, \{a, b\}\}$. Then $G = \{c\}$ is a weakly $\mathcal{I}_g$-closed set but not $\mathcal{I}_g$-closed.

Theorem 3.6. *Let $(X, \tau, \mathcal{I})$ be an ideal topological space and $G \subseteq X$. The following properties are equivalent:*

(1). G is a mildly $\mathcal{I}_g$-$\star$-closed set,

(2). cl(int(G)) $\subseteq$ H whenever $G \subseteq H$ and H is a $\star$-g-open set in X.*

Proof. (1) $\Rightarrow$ (2) : Let G be a mildly $\mathcal{I}_g$-$\star$-closed set in $(X, \tau, \mathcal{I})$. Suppose that G $\subseteq$ H and H is a $\star$-g-open set in X. We have (int(G))* $\subseteq$ H. Since int(G) $\subseteq$ G $\subseteq$ H, then (int(G))* $\cup$ int(G) $\subseteq$ H. This implies that cl*(int(G)) $\subseteq$ H.
(2) $\Rightarrow$ (1) : Let cl*(int(G)) $\subseteq$ H whenever G $\subseteq$ H and H is $\star$-g-open in X. Since (int(G))* $\cup$ int(G) $\subseteq$ H, then (int(G))* $\subseteq$ H whenever G $\subseteq$ H and H is a $\star$-g-open set in X. Therefore G is a mildly $\mathcal{I}_g$-$\star$-closed set in $(X, \tau, \mathcal{I})$. □

Theorem 3.7. *Let $(X, \tau, \mathcal{I})$ be an ideal topological space and $G \subseteq X$. If G is $\star$-g-open and mildly $\mathcal{I}_g$-$\star$-closed, then G is $pre^*_{\mathcal{I}}$-closed.*

Proof. Let G be $\star$-g-open and mildly $\mathcal{I}_g$-$\star$-closed in $(X, \tau, \mathcal{I})$. Since G is $\star$-g-open and mildly $\mathcal{I}_g$-$\star$-closed, cl*(int(G)) $\subseteq$ G by Theorem 3.6. Thus, G is a $pre^*_{\mathcal{I}}$-closed set in $(X, \tau, \mathcal{I})$. □

Theorem 3.8. *Let $(X, \tau, \mathcal{I})$ be an ideal topological space and $G \subseteq X$. If G is a mildly $\mathcal{I}_g$-$\star$-closed set, then (int(G))* $\backslash$G contains no any nonempty $\star$-g-closed set.*

Proof. Let G be a mildly $\mathcal{I}_g$-$\star$-closed set in $(X, \tau, \mathcal{I})$. Suppose that H is a $\star$-g-closed set such that H $\subseteq$ (int(G))*$\backslash$G. Since G is a mildly $\mathcal{I}_g$-$\star$-closed set, X$\backslash$H is $\star$-g-open and G $\subseteq$ X$\backslash$H, then (int(G))* $\subseteq$ X$\backslash$H. We have H $\subseteq$ X$\backslash$(int(G))*. Hence, H $\subseteq$ (int(G))* $\cap$ (X$\backslash$(int(G))*) = $\emptyset$. Thus, (int(G))*$\backslash$G contains no any nonempty $\star$-g-closed set. □

Theorem 3.9. *Let $(X, \tau, \mathcal{I})$ be an ideal topological space and $G \subseteq X$. If G is a mildly $\mathcal{I}_g$-$\star$-closed set, then cl*(int(G))$\backslash$G contains no any nonempty $\star$-g-closed set.*

Proof. Suppose that H is a $\star$-g-closed set such that H $\subseteq$ cl*(int(G))\G. By Theorem 3.8, it follows from the fact that cl*(int(G))\G = ((int(G))* $\cup$ int(G))\G. □

Theorem 3.10. *Let (X, τ, $\mathcal{I}$) be an ideal topological space. The following properties are equivalent:*

(1). G is $pre^*_{\mathcal{I}}$-closed for each mildly $\mathcal{I}_g$-$\star$-closed set G in (X, τ, $\mathcal{I}$),

(2). Each singleton $\{x\}$ of X is a $\star$-g-closed set or $\{x\}$ is $pre^*_{\mathcal{I}}$-open.

Proof. (1) $\Rightarrow$ (2) : Let G be $pre^*_{\mathcal{I}}$-closed for each mildly $\mathcal{I}_g$-$\star$-closed set G in (X, τ, $\mathcal{I}$) and x $\in$ X. We have cl*(int(G)) $\subseteq$ G for each mildly $\mathcal{I}_g$-$\star$-closed set G in (X, τ, $\mathcal{I}$). Assume that {x} is not a $\star$-g-closed set. It follows that X is the only $\star$-g-open set containing X\{x}. Then, X\{x} is a mildly $\mathcal{I}_g$-$\star$-closed set in (X, τ, $\mathcal{I}$). Thus, cl*(int(X\{x})) $\subseteq$ X\{x} and hence {x} $\subseteq$ int*(cl({x})). Consequently, {x} is $pre^*_{\mathcal{I}}$-open.

(2) $\Rightarrow$ (1) : Let G be a mildly $\mathcal{I}_g$-$\star$-closed set in (X, τ, $\mathcal{I}$). Let x $\in$ cl*(int(G)).

Suppose that {x} is $pre^*_{\mathcal{I}}$-open. We have {x} $\subseteq$ int*(cl({x})). Since x $\in$ cl*(int(G)), then int*(cl({x})) $\cap$ int(G) $\neq \emptyset$. It follows that cl({x}) $\cap$ int(G) $\neq \emptyset$. We have cl({x} $\cap$ int(G)) $\neq \emptyset$ and then {x} $\cap$ int(G) $\neq \emptyset$. Hence, x $\in$ int(G). Thus, we have x $\in$ G.

Suppose that {x} is a $\star$-g-closed set. By Theorem 3.9, cl*(int(G))\G does not contain {x}. Since x $\in$ cl*(int(G)), then we have x $\in$ G. Consequently, we have x $\in$ G. Thus, cl*(int(G)) $\subseteq$ G and hence G is $pre^*_{\mathcal{I}}$-closed. □

Theorem 3.11. *Let (X, τ, $\mathcal{I}$) be an ideal topological space and G $\subseteq$ X. If G is a mildly $\mathcal{I}_g$-$\star$-closed set, then int(G) = H\K where H is $\mathcal{I}$-R closed and K contains no any nonempty $\star$-g-closed set.*

Proof. Let G be a mildly $\mathcal{I}_g$-$\star$-closed set in (X, τ, $\mathcal{I}$). Take K = (int(G))*\G. Then, by Theorem 3.8, K contains no any nonempty $\star$-g-closed set. Take H = cl*(int(G)). Then H = cl*(int(H)). Moreover, we have H\K = ((int(G))* $\cup$ int(G))\((int(G))*\G) = ((int(G))* $\cup$ int(G)) $\cap$ (X\(int(G))* $\cup$ G) = int(G). □

Theorem 3.12. *Let (X, τ, $\mathcal{I}$) be an ideal topological space and G $\subseteq$ X. Assume that G is a mildly $\mathcal{I}_g$-$\star$-closed set. The following properties are equivalent:*

(1). G is $pre^*_{\mathcal{I}}$-closed,

(2). cl(int(G))\G is a $\star$-g-closed set,*

(3). (int(G)) \G is a $\star$-g-closed set.*

Proof. (1) $\Rightarrow$ (2) : Let G be $pre^*_{\mathcal{I}}$-closed. We have cl*(int(G)) $\subseteq$ G. Then, cl*(int(G))\G = $\emptyset$. Thus, cl*(int(G))\G is a $\star$-g-closed set.

(2) $\Rightarrow$ (1) : Let cl*(int(G))\G be a $\star$-g-closed set. Since G is a mildly $\mathcal{I}_g$-$\star$-closed set in (X, τ, $\mathcal{I}$), then by Theorem 3.9, cl*(int(G))\G = $\emptyset$. Hence, we have cl*(int(G)) $\subseteq$ G. Thus, G is $pre^*_{\mathcal{I}}$-closed.

(2) $\Leftrightarrow$ (3) : It follows easily from that cl*(int(G))\G = (int(G))*\G. □

Theorem 3.13. *Let (X, τ, $\mathcal{I}$) be an ideal topological space and G $\subseteq$ X be a mildly $\mathcal{I}_g$-$\star$-closed set. Then G $\cup$ (X\(int(G))*) is a mildly $\mathcal{I}_g$-$\star$-closed set in (X, τ, $\mathcal{I}$).*

Proof. Let G be a mildly $\mathcal{I}_g$-$\star$-closed set in (X, τ, $\mathcal{I}$). Suppose that H is a $\star$-g-open set such that G $\cup$ (X\(int(G))*) $\subseteq$ H. We have X\H $\subseteq$ X\(G $\cup$ (X\(int(G))*)) = (X\G) $\cap$ (int(G))* = (int(G))*\G. Since X\H is a $\star$-g-closed set and G is a mildly $\mathcal{I}_g$-$\star$-closed set, it follows from Theorem 3.8 that X\H = $\emptyset$. Hence, X = H. Thus, X is the only $\star$-g-open set containing G $\cup$ (X\(int(G))*). Consequently, G $\cup$ (X\(int(G))*) is a mildly $\mathcal{I}_g$-$\star$-closed set in (X, τ, $\mathcal{I}$). □

Corollary 3.14. *Let $(X, \tau, \mathcal{I})$ be an ideal topological space and $G \subseteq X$ be a mildly $\mathcal{I}_g$-$\star$-closed set. Then $(int(G))^*\backslash G$ is a mildly $\mathcal{I}_g$-$\star$-open set in $(X, \tau, \mathcal{I})$.*

Proof. Since $X\backslash((int(G))^*\backslash G) = G \cup (X\backslash(int(G))^*)$, it follows from Theorem 3.13 that $(int(G))^*\backslash G$ is a mildly $\mathcal{I}_g$-$\star$-open set in $(X, \tau, \mathcal{I})$. □

Theorem 3.15. *Let $(X, \tau, \mathcal{I})$ be an ideal topological space and $G \subseteq X$. The following properties are equivalent:*

(1). G is a $\star$-closed and open set,

(2). G is $\mathcal{I}$-R closed and open set,

(3). G is a mildly $\mathcal{I}_g$-$\star$-closed and open set.

Proof. (1) $\Rightarrow$ (2) $\Rightarrow$ (3) : Obvious.

(3) $\Rightarrow$ (1) : Since G is open and mildly $\mathcal{I}_g$-$\star$-closed, $cl^*(int(G)) \subseteq G$ and so $G = cl^*(int(G))$. Then G is $\mathcal{I}$-R closed and hence it is $\star$-closed. □

Proposition 3.16. *Every $pre^*_{\mathcal{I}}$-closed set is mildly $\mathcal{I}_g$-$\star$-closed.*

Proof. Let $G \subseteq H$ and H be a $\star$-g-open set in X. Since G is $pre^*_{\mathcal{I}}$-closed, $cl^*(int(G)) \subseteq G \subseteq H$. Hence G is a mildly $\mathcal{I}_g$-$\star$-closed set by Theorem 3.6. □

Example 3.17. *Let $(X, \tau, \mathcal{I})$ be an ideal topological space such that $X = \{a, b, c, d\}$, $\tau = \{\emptyset, \{a\}, \{d\}, \{a, d\}, \{b, d\}, \{a, b, d\}, X\}$ and $\mathcal{I} = \{\emptyset\}$. Then $\{c, d\}$ is a mildly $\mathcal{I}_g$-$\star$-closed set but not $pre^*_{\mathcal{I}}$-closed.*

4. Further Properties

Theorem 4.1. *Let $(X, \tau, \mathcal{I})$ be an ideal topological space. The following properties are equivalent:*

(1). Each subset of $(X, \tau, \mathcal{I})$ is a mildly $\mathcal{I}_g$-$\star$-closed set,

(2). G is $pre^*_{\mathcal{I}}$-closed for each $\star$-g-open set G in X.

Proof. (1) $\Rightarrow$ (2) : Suppose that each subset of $(X, \tau, \mathcal{I})$ is a mildly $\mathcal{I}_g$-$\star$-closed set. Let G be a $\star$-g-open set. Since G is mildly $\mathcal{I}_g$-$\star$-closed, then we have $cl^*(int(G)) \subseteq G$. Thus, G is $pre^*_{\mathcal{I}}$-closed.

(2) $\Rightarrow$ (1) : Let H be a subset of $(X, \tau, \mathcal{I})$ and G be a $\star$-g-open set such that $H \subseteq G$. By (2), we have $cl^*(int(H)) \subseteq cl^*(int(G)) \subseteq G$. Thus, H is a mildly $\mathcal{I}_g$-$\star$-closed set in $(X, \tau, \mathcal{I})$. □

Theorem 4.2. *Let $(X, \tau, \mathcal{I})$ be an ideal topological space. If G is a mildly $\mathcal{I}_g$-$\star$-closed set and $G \subseteq H \subseteq cl^*(int(G))$, then H is a mildly $\mathcal{I}_g$-$\star$-closed set.*

Proof. Let $H \subseteq K$ and K be a $\star$-g-open set in X. Since $G \subseteq K$ and G is a mildly $\mathcal{I}_g$-$\star$-closed set, then $cl^*(int(G)) \subseteq K$. Since $H \subseteq cl^*(int(G))$, then $cl^*(int(H)) \subseteq cl^*(int(G)) \subseteq K$. Thus, $cl^*(int(H)) \subseteq K$ and hence, H is a mildly $\mathcal{I}_g$-$\star$-closed set. □

Corollary 4.3. *Let $(X, \tau, \mathcal{I})$ be an ideal topological space. If G is a mildly $\mathcal{I}_g$-$\star$-closed and open set, then $cl^*(G)$ is a mildly $\mathcal{I}_g$-$\star$-closed set.*

Proof. Let G be a mildly $\mathcal{I}_g$-$\star$-closed and open set in $(X, \tau, \mathcal{I})$. We have $G \subseteq cl^*(G) \subseteq cl^*(G) = cl^*(int(G))$. Hence, by Theorem 4.2, $cl^*(G)$ is a mildly $\mathcal{I}_g$-$\star$-closed set in $(X, \tau, \mathcal{I})$. □

Theorem 4.4. *Let (X, τ, $\mathcal{I}$) be an ideal topological space and $G \subseteq X$. If G is a nowhere dense set, then G is a mildly $\mathcal{I}_g$-⋆-closed set.*

Proof. Let G be a nowhere dense set in X. Since int(G) ⊆ int(cl(G)), then int(G) = ∅. Hence, cl*(int(G)) = ∅. Thus, G is a mildly $\mathcal{I}_g$-⋆-closed set in (X, τ, $\mathcal{I}$). □

Remark 4.5. *The reverse of Theorem 4.4 is not true in general as shown in the following Example.*

Example 4.6. *Let X, τ and $\mathcal{I}$ be as in Example 3.5(b). Then {a} is a mildly $\mathcal{I}_g$-⋆-closed set but not a nowhere dense set.*

Remark 4.7. *The union of two mildly $\mathcal{I}_g$-⋆-closed sets in an ideal topological space need not be a mildly $\mathcal{I}_g$-⋆-closed set.*

Example 4.8. *Let X, τ and $\mathcal{I}$ be as in Example 3.5(b). We have A = {a} and B = {b, c} are mildly $\mathcal{I}_g$-⋆-closed. But their union A∪B = {a, b, c} is not a mildly $\mathcal{I}_g$-⋆-closed set.*

Theorem 4.9. *Let (X, τ, $\mathcal{I}$) be an ideal topological space and G ⊆ X. Then G is a mildly $\mathcal{I}_g$-⋆-open set if and only if H ⊆ int*(cl(G)) whenever H ⊆ G and H is a ⋆-g-closed set.*

Proof. Let H be a ⋆-g-closed set in X and H ⊆ G. It follows that X\H is a ⋆-g-open set and X\G ⊆ X\H. Since X\G is a mildly $\mathcal{I}_g$-⋆-closed set, then cl*(int(X\G)) ⊆ X\H. We have X\int*(cl(G)) ⊆ X\H. Thus, H ⊆ int*(cl(G)).
Conversely, let K be a ⋆-g-open set in X and X\G ⊆ K. Since X\K is a ⋆-g-closed set such that X\K ⊆ G, then X\K ⊆ int*(cl(G)). We have X\int*(cl(G)) = cl*(int(X\G)) ⊆ K. Thus, X\G is a mildly $\mathcal{I}_g$-⋆-closed set. Hence, G is a mildly $\mathcal{I}_g$-⋆-open set in (X, τ, $\mathcal{I}$). □

Theorem 4.10. *Let (X, τ, $\mathcal{I}$) be an ideal topological space and G ⊆ X. If G is a mildly $\mathcal{I}_g$-⋆-closed set, then cl*(int(G))\G is a mildly $\mathcal{I}_g$-⋆-open set in (X, τ, $\mathcal{I}$).*

Proof. Let G be a mildly $\mathcal{I}_g$-⋆-closed set in (X, τ, $\mathcal{I}$). Suppose that H is a ⋆-g-closed set such that H ⊆ cl*(int(G))\G. Since G is a mildly $\mathcal{I}_g$-⋆-closed set, it follows from Theorem 3.9 that H = ∅. Thus, we have H ⊆ int*(cl(cl*(int(G))\G)). It follows from Theorem 4.9 that cl*(int(G))\G is a mildly $\mathcal{I}_g$-⋆-open set in (X, τ, $\mathcal{I}$). □

Theorem 4.11. *Let (X, τ, $\mathcal{I}$) be an ideal topological space and G ⊆ X. If G is a mildly $\mathcal{I}_g$-⋆-open set, then H = X whenever H is a ⋆-g-open set and int*(cl(G)) ∪ (X\G) ⊆ H.*

Proof. Let H be a ⋆-g-open set in X and int*(cl(G)) ∪ (X\G) ⊆ H. We have X\H ⊆ (X\int*(cl(G))) ∩ G = cl*(int(X\G))\(X\G). Since X\H is a ⋆-g-closed set and X\G is a mildly $\mathcal{I}_g$-⋆-closed set, it follows from Theorem 3.9 that X\H = ∅. Thus, we have H = X. □

Theorem 4.12. *Let (X, τ, $\mathcal{I}$) be an ideal topological space. If G is a mildly $\mathcal{I}_g$-⋆-open set and int*(cl(G)) ⊆ H ⊆ G, then H is a mildly $\mathcal{I}_g$-⋆-open set.*

Proof. Let G be a mildly $\mathcal{I}_g$-⋆-open set and int*(cl(G)) ⊆ H ⊆ G. Since int*(cl(G)) ⊆ H ⊆ G, then int*(cl(G)) = int*(cl(H)). Let K be a ⋆-g-closed set and K ⊆ H. We have K ⊆ G. Since G is a mildly $\mathcal{I}_g$-⋆-open set, it follows from Theorem 4.9 that K ⊆ int*(cl(G)) = int*(cl(H)). Hence, by Theorem 4.9, H is a mildly $\mathcal{I}_g$-⋆-open set in (X, τ, $\mathcal{I}$). □

Corollary 4.13. *Let (X, τ, $\mathcal{I}$) be an ideal topological space and G ⊆ X. If G is a mildly $\mathcal{I}_g$-⋆-open and closed set, then int*(G) is a mildly $\mathcal{I}_g$-⋆-open set.*

Proof. Let G be a mildly $\mathcal{I}_g$-⋆-open and closed set in (X, τ, $\mathcal{I}$). Then int*(cl(G)) = int*(G) ⊆ int*(G) ⊆ G. Thus, by Theorem 4.12, int*(G) is a mildly $\mathcal{I}_g$-⋆-open set in (X, τ, $\mathcal{I}$). □

Definition 4.14. *A subset H of an ideal topological space (X, τ, $\mathcal{I}$) is called $S_{\mathcal{I}}$-set if H = M$\cup$N where M is $\star$-g-closed and N is pre$^*_{\mathcal{I}}$-open.*

Remark 4.15. *Every pre$^*_{\mathcal{I}}$-open (resp. $\star$-g-closed) set is an $S_{\mathcal{I}}$-set but not conversely.*

Example 4.16. *Let (X, τ, $\mathcal{I}$) be an ideal topological space such that X = $\{a, b, c\}$, τ = $\{\emptyset, \{a\}, \{b\}, \{a, b\}, X\}$ and $\mathcal{I}$ = $\{\emptyset, \{a\}\}$. Then $\{c\}$ is an $S_{\mathcal{I}}$-set but not pre$^*_{\mathcal{I}}$-open. Also $\{a, b\}$ is an $S_{\mathcal{I}}$-set but not $\star$-g-closed.*

Theorem 4.17. *For a subset H of (X, τ, $\mathcal{I}$), the following are equivalent.*

(1). H is pre$^*_{\mathcal{I}}$-open.

(2). H is an $S_{\mathcal{I}}$-set and mildly $\mathcal{I}_g$-$\star$-open.

Proof. (1) $\Rightarrow$ (2): By Remark 4.15, H is an $S_{\mathcal{I}}$-set. By Proposition 3.16, H is mildly $\mathcal{I}_g$-$\star$-open.
(2) $\Rightarrow$ (1): Let H be an $S_{\mathcal{I}}$-set and mildly $\mathcal{I}_g$-$\star$-open. Then there exist a $\star$-g-closed set M and a $pre^*_{\mathcal{I}}$-open set N such that H =M$\cup$N. Since M $\subseteq$ H and H is mildly $\mathcal{I}_g$-$\star$-open, by Theorem 4.9, M $\subseteq$ int*(cl(H)). Also, we have N $\subseteq$ int*(cl(N)). Since N $\subseteq$ H, N $\subseteq$ int*(cl(N)) $\subseteq$ int*(cl(H)). Then H = M$\cup$N $\subseteq$ int*(cl(H)). So H is $pre^*_{\mathcal{I}}$-open. □

The following Examples show that the concepts of mildly $\mathcal{I}_g$-$\star$-open set and $S_{\mathcal{I}}$-set are independent.

Example 4.18. *Let X, τ and $\mathcal{I}$ be as in Example 4.16. Then $\{a, c\}$ is an $S_{\mathcal{I}}$-set but not mildly $\mathcal{I}_g$-$\star$-open.*

Example 4.19. *Let (X, τ, $\mathcal{I}$) be an ideal topological space such that X = $\{a, b, c, d\}$, τ = $\{\emptyset, \{a\}, \{c\}, \{a, c\}, \{c, d\}, \{a, c, d\}, \{b, c, d\}, X\}$ and $\mathcal{I}$ = $\{\emptyset\}$. Then $\{d\}$ is a mildly $\mathcal{I}_g$-$\star$-open set but not an $S_{\mathcal{I}}$-set.*

5. $\star$-g-$pre^*_{\mathcal{I}}$-normal Spaces

Definition 5.1. *An ideal topological space (X, τ, $\mathcal{I}$) is said to be $\star$-g-pre$^*_{\mathcal{I}}$-normal if for every pair of disjoint $\star$-g-closed subsets A, B of X, there exist disjoint pre$^*_{\mathcal{I}}$-open sets U, V of X such that A $\subseteq$ U and B $\subseteq$ V.*

Theorem 5.2. *The following properties are equivalent for a space (X, τ, $\mathcal{I}$).*

(1). X is $\star$-g-pre$^*_{\mathcal{I}}$-normal;

(2). for any disjoint $\star$-g-closed sets A and B, there exist disjoint mildly $\mathcal{I}_g$-$\star$-open sets U, V of X such that A $\subseteq$ U and B $\subseteq$ V;

(3). for any $\star$-g-closed set A and any $\star$-g-open set B containing A, there exists a mildly $\mathcal{I}_g$-$\star$-open set U such that A $\subseteq$ U $\subseteq$ cl(int(U)) $\subseteq$ B.*

Proof. (1) $\Rightarrow$ (2): The proof is obvious.
(2) $\Rightarrow$ (3): Let A be any $\star$-g-closed set of X and B any $\star$-g-open set of X such that A $\subseteq$ B. Then A and X\B are disjoint $\star$-g-closed sets of X. By (2), there exist disjoint mildly $\mathcal{I}_g$-$\star$-open sets U, V of X such that A $\subseteq$ U and X\B $\subseteq$ V. Since V is a mildly $\mathcal{I}_g$-$\star$-open set, by Theorem 4.9, X\B $\subseteq$ int*(cl(V)) and U$\cap$int*(cl(V)) = $\emptyset$. Therefore we obtain cl*(int(U)) $\subseteq$ cl*(int(X\V)) and hence A $\subseteq$ U $\subseteq$ cl*(int(U)) $\subseteq$ B.
(3) $\Rightarrow$ (1): Let A and B be any disjoint $\star$-g-closed sets of X. Then A $\subseteq$ X\B and X\B is $\star$-g-open and hence there exists a mildly $\mathcal{I}_g$-$\star$-open set G of X such that A $\subseteq$ G $\subseteq$ cl*(int(G)) $\subseteq$ X\B. Put U = int*(cl(G)) and V = X\cl*(int(G)). Then U and V are disjoint $pre^*_{\mathcal{I}}$-open sets of X such that A $\subseteq$ U and B $\subseteq$ V. Therefore X is $\star$-g-$pre^*_{\mathcal{I}}$-normal. □

Definition 5.3. *A function $f : (X, \tau, \mathcal{I}) \to (Y, \sigma)$ is said to be mildly $\mathcal{I}_g$-$\star$-continuous if $f^{-1}(V)$ is mildly $\mathcal{I}_g$-$\star$-closed in X for every closed set V of Y.*

Definition 5.4. *A function $f : (X, \tau, \mathcal{I}) \to (Y, \sigma, \mathcal{J})$ is called mildly $(\mathcal{I}, \mathcal{J})_g$-irresolute if $f^{-1}(V)$ is mildly $\mathcal{I}_g$-$\star$-closed in X for every mildly $\mathcal{J}_g$-$\star$-closed of Y.*

Definition 5.5. *A function $f : (X, \tau, \mathcal{I}) \to (Y, \sigma, \mathcal{J})$ is said to be pre-$\star$-g-closed if f(V) is $\star$-g-closed in Y for every $\star$-g-closed set V of X.*

Theorem 5.6. *Let $f : (X, \tau, \mathcal{I}) \to (Y, \sigma, \mathcal{J})$ be a mildly $\mathcal{I}_g$-$\star$-continuous pre-$\star$-g-closed injection. If Y is $\star$-g-normal, then X is $\star$-g-pre$^*_{\mathcal{I}}$-normal.*

Proof. Let A and B be disjoint $\star$-g-closed sets of X. Since f is pre-$\star$-g-closed injection, f(A) and f(B) are disjoint $\star$-g-closed sets of Y. By the $\star$-g-normality of Y, there exist disjoint open sets U and V of Y such that f(A) $\subseteq$ U and f(B) $\subseteq$ V. Since f is mildly $\mathcal{I}_g$-$\star$-continuous, then $f^{-1}(U)$ and $f^{-1}(V)$ are disjoint mildly $\mathcal{I}_g$-$\star$-open sets such that $A \subseteq f^{-1}(U)$ and $B \subseteq f^{-1}(V)$. Therefore X is $\star$-g-$pre^*_{\mathcal{I}}$-normal by Theorem 5.2. □

Theorem 5.7. *Let $f : (X, \tau, \mathcal{I}) \to (Y, \sigma, \mathcal{J})$ be a mildly $(\mathcal{I}, \mathcal{J})_g$-$\star$-irresolute pre-$\star$-g-closed injection. If Y is $\star$-g-pre$^*_{\mathcal{I}}$-normal, then X is $\star$-g-pre$^*_{\mathcal{I}}$-normal.*

Proof. Let A and B be disjoint $\star$-g-closed sets of X. Since f is pre-$\star$-g-closed injection, f(A) and f(B) are disjoint $\star$-g-closed sets of Y. Since Y is $\star$-g-$pre^*_{\mathcal{I}}$-normal, by Theorem 5.2, there exist disjoint mildly $\mathcal{J}_g$-$\star$-open sets U and V of Y such that f(A) $\subseteq$ U and f(B) $\subseteq$ V. Since f is mildly $(\mathcal{I}, \mathcal{J})_g$-$\star$-irresolute, then $f^{-1}(U)$ and $f^{-1}(V)$ are disjoint mildly $\mathcal{I}_g$-$\star$-open sets of X such that $A \subseteq f^{-1}(U)$ and $B \subseteq f^{-1}(V)$. Therefore X is $\star$-g-$pre^*_{\mathcal{I}}$-normal. □

References

[1] A.Acikgoz and S.Yuksel, *Some new sets and decompositions of $A_{\mathcal{I}-R}$-continuity, α-$\mathcal{I}$-continuity, continuity via idealization*, Acta Math. Hungar., 114(1-2)(2007), 79-89.

[2] J.Dontchev, M.Ganster and T.Noiri, *Unified operation approach of generalized closed sets via topological ideals*, Math. Japonica, 49(1999), 395-401.

[3] E.Ekici, *On $\mathcal{AC}_{\mathcal{I}}$-sets, $\mathcal{BC}_{\mathcal{I}}$-sets, $\beta^*_{\mathcal{I}}$-open sets and decompositions of continuity in ideal topological spaces*, Creat. Math. Inform, 20(2011), 47-54.

[4] E.Ekici and S.Ozen, *A generalized class of τ^* in ideal spaces*, Filomat, 27(4)(2013), 529-535.

[5] D.Jankovic and T.R.Hamlett, *New topologies from old via ideals*, Amer. Math. Monthly, 97(4)(1990), 295-310.

[6] K.Kuratowski, *Topology*, Vol. I, Academic Press, New York, (1966).

[7] N.Levine, *Generalized closed sets in topology*, Rend. Cir. Math. Palermo, 19(2)(1970), 89-96.

[8] D.Mandal and M.N.Mukherjee, *Certain new classes of generalized closed sets and their applications in ideal topological spaces*, Filomat, 29(5)(2015), 1113-1120.

[9] M.Navaneethakrishnan and J.Paulraj Joseph, *g-closed sets in ideal topological spaces*, Acta Math. Hungar., 119(4)(2008), 365-371.

[10] J.K.Park and J.H.Park, *Mildly generalized closed sets*, almost normal and mildly normal spaces, Chaos, Solitons and Fractals, 20(2004), 1103-1111.

[11] O.Ravi, R.Senthil Kumar and A.Hamari Choudhi, *Weakly $\mathcal{I}_g$-closed sets*, Bulletin of the international mathematical virtual institute, 4(2014), 1-9.

[12] P.Sundaram and N.Nagaveni, *On weakly generalized continuous maps, weakly generalized closed maps and weakly generalized irresolute maps in topological spaces*, Far East J. Math. Sci., 6(6)(1998), 903-1012.

[13] P.Sundaram and A.Pushpalatha, *Strongly generalized closed sets in topological spaces*, Far East J. Math. Sci., 3(4)(2001), 563-575.

[14] R.Vaidyanathaswamy, *Set Topology*, Chelsea Publishing Company, (1946).