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Weakly $\mathcal{I}_g$-⋆-closed Sets

Research Article

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Abstract: In this paper, the notion of weakly $\mathcal{I}_g$-⋆-closed sets is introduced and studied in ideal topological spaces. The relationships of weakly $\mathcal{I}_g$-⋆-closed sets with various like sets are investigated.

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1. Introduction

The first step of generalizing closed sets was done by Levine in 1970 [6]. He defined a subset G of a topological space (X, τ) to be g-closed if its closure belongs to every open superset of G. As the weak form of g-closed sets, the notion of weakly g-closed sets was introduced and studied by Sundaram and Nagaveni [10]. Sundaram and Pushpalatha [11] introduced and studied the notion of strongly g-closed sets, which is implied by that of closed sets and implies that of g-closed sets. Park and Park [8] introduced and studied mildly g-closed sets, which is properly placed between the classes of strongly g-closed and weakly g-closed sets. Moreover, the relations with other notions directly or indirectly connected with g-closed were investigated by them. In 2013, Ekici and Ozen [3] introduced a generalized class of τ^* in ideal topological spaces. In 2015, Mandal and Mukherjee [7] introduced and studied the notions of ⋆-g-closed and ⋆-g-open sets in ideal topological spaces. The main aim of this paper is to study the notion of weakly $\mathcal{I}_g$-⋆-closed sets in ideal topological spaces. The relationships of weakly $\mathcal{I}_g$-⋆-closed sets with various like sets are discussed.

2. Preliminaries

In this paper, (X, τ) represents topological space on which no separation axioms are assumed unless explicitly stated. The closure and the interior of a subset H of a topological space (X, τ) will be denoted by cl(H) and int(H), respectively. An ideal $\mathcal{I}$ on a topological space (X, τ) is a nonempty collection of subsets of X which satisfies

(1). P $\in \mathcal{I}$ and Q $\subseteq$ P imply Q $\in \mathcal{I}$ and

(2). P $\in \mathcal{I}$ and Q $\in \mathcal{I}$ imply P$\cup$Q $\in \mathcal{I}$ [5].

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Given a topological space (X, τ) with an ideal $\mathcal{I}$ on X if $\mathcal{P}$(X) is the set of all subsets of X, a set operator $(\bullet)^\star : \mathcal{P}(X) \to \mathcal{P}(X)$, called a local function [5] of Z with respect to τ and $\mathcal{I}$ is defined as follows: for Z $\subseteq$ X, $Z^\star(\mathcal{I}, \tau) = \{x \in X \mid U \cap Z \notin \mathcal{I}$ for every $U \in \tau(x)\}$ where $\tau(x) = \{U \in \tau \mid x \in U\}$. A Kuratowski closure operator $cl^\star(\bullet)$ for a topology $\tau^\star(\mathcal{I}, \tau)$, called the $\star$-topology and finer than τ, is defined by $cl^\star(Z) = Z \cup Z^\star(\mathcal{I}, \tau)$ [12]. We will simply write $Z^\star$ for $Z^\star(\mathcal{I}, \tau)$ and $\tau^\star$ for $\tau^\star(\mathcal{I}, \tau)$. If $\mathcal{I}$ is an ideal on X, then (X, τ, $\mathcal{I}$) is called an ideal topological space. On the other hand, (A, τ_A, $\mathcal{I}_A$) where τ_A is the relative topology on A and $\mathcal{I}_A = \{A \cap J : J \in \mathcal{I}\}$ is an ideal topological space for an ideal topological space (X, τ, $\mathcal{I}$) and A $\subseteq$ X [4]. For a subset Z $\subseteq$ X, cl*(Z) and int*(Z) will, respectively, denote the closure and the interior of Z in (X, τ*).

Definition 2.1 ([7]). *A subset G of an ideal topological space $(X, \tau, \mathcal{I})$ is said to be*

(1). $\star$-g-closed if cl(G) $\subseteq$ H whenever G $\subseteq$ H and H is $\star$-open in X;

(2). $\star$-g-open if X$\setminus$G is $\star$-g-closed.

Definition 2.2. *A subset G of an ideal topological space (X, τ, $\mathcal{I}$) is said to be*

(1). $pre^*_{\mathcal{I}}$-open [2] if G $\subseteq$ int(cl(G)).*

(2). $pre^*_{\mathcal{I}}$-closed [2] if X$\setminus$G is $pre^*_{\mathcal{I}}$-open (or) $cl^\star(int(G)) \subseteq G$.

(3). $\mathcal{I}$-R closed [1] if G = cl(int(G)).*

(4). $\star$-closed [4] if G = cl(G) or G* $\subseteq$ G.*

Remark 2.3 ([3]). *In any ideal topological space, every $\mathcal{I}$-R closed set is $\star$-closed but not conversely.*

Definition 2.4. *An ideal topological space $(X, \tau, \mathcal{I})$ is said to be $\star$-normal [9] if for any two disjoint closed sets A and B of X, there exist disjoint $\star$-open sets U, V such that $A \subseteq U$ and $B \subseteq V$.*

3. Properties of Weakly $\mathcal{I}_g$-$\star$-closed Sets

Definition 3.1. *Let $(X, \tau, \mathcal{I})$ be an ideal topological space. A subset G of $(X, \tau, \mathcal{I})$ is said to be weakly $\mathcal{I}_g$-$\star$-closed if $(int(G))^\star \subseteq H$ whenever $G \subseteq H$ and H is an $\star$-open set in X.*

Example 3.2. *Let (X, τ, $\mathcal{I}$) be an ideal topological space such that X = $\{a, b, c\}$, $\tau = \{\emptyset, \{a\}, X\}$ and $\mathcal{I} = \{\emptyset\}$. Then $\{b\}$ is a weakly $\mathcal{I}_g$-$\star$-closed set but $\{a\}$ is not a weakly $\mathcal{I}_g$-$\star$-closed set.*

Definition 3.3. *Let (X, τ, $\mathcal{I}$) be an ideal topological space and G $\subseteq$ X. Then G is said to be a weakly $\mathcal{I}_g$-$\star$-open set if X$\setminus$G is a weakly $\mathcal{I}_g$-$\star$-closed set.*

Theorem 3.4. *Let (X, τ, $\mathcal{I}$) be an ideal topological space and G $\subseteq$ X. The following properties are equivalent:*

(1). G is a weakly $\mathcal{I}_g$-$\star$-closed set,

(2). cl(int(G)) $\subseteq$ H whenever G $\subseteq$ H and H is an $\star$-open set in X.*

Proof. (1) $\Rightarrow$ (2): Let G be a weakly $\mathcal{I}_g$-$\star$-closed set in $(X, \tau, \mathcal{I})$. Suppose that $G \subseteq H$ and H is an $\star$-open set in X. We have $(int(G))^\star \subseteq H$. Since $int(G) \subseteq G \subseteq H$, then $(int(G))^\star \cup int(G) \subseteq H$. This implies that $cl^\star(int(G)) \subseteq H$.
(2) $\Rightarrow$ (1): Let $cl^\star(int(G)) \subseteq H$ whenever $G \subseteq H$ and H is $\star$-open in X. Since $(int(G))^\star \cup int(G) \subseteq H$, then $(int(G))^\star \subseteq H$ whenever $G \subseteq H$ and H is an $\star$-open set in X. Therefore G is a weakly $\mathcal{I}_g$-$\star$-closed set in $(X, \tau, \mathcal{I})$. □

Theorem 3.5. *Let $(X, \tau, \mathcal{I})$ be an ideal topological space and $G \subseteq X$. If G is $\star$-open and weakly $\mathcal{I}_g$-$\star$-closed, then G is $pre^*_{\mathcal{I}}$-closed.*

Proof. Let G be an $\star$-open and weakly $\mathcal{I}_g$-$\star$-closed set in $(X, \tau, \mathcal{I})$. Since G is $\star$-open and weakly $\mathcal{I}_g$-$\star$-closed, cl*(int(G)) $\subseteq$ G by Theorem 3.4. Thus, G is a $pre^*_{\mathcal{I}}$-closed set in $(X, \tau, \mathcal{I})$. □

Theorem 3.6. *Let $(X, \tau, \mathcal{I})$ be an ideal topological space and $G \subseteq X$. If G is a weakly $\mathcal{I}_g$-$\star$-closed set, then $(int(G))^* \setminus G$ contains no any nonempty $\star$-closed set.*

Proof. Let G be a weakly $\mathcal{I}_g$-$\star$-closed set in $(X, \tau, \mathcal{I})$. Suppose that H is a $\star$-closed set such that H $\subseteq$ (int(G))*$\setminus$G. Since G is a weakly $\mathcal{I}_g$-$\star$-closed set, X$\setminus$H is $\star$-open and G $\subseteq$ X$\setminus$H, then (int(G))* $\subseteq$ X$\setminus$H. We have H $\subseteq$ X$\setminus$(int(G))*. Hence, H $\subseteq$ (int(G))* $\cap$ (X$\setminus$(int(G))*) = $\emptyset$. Thus, (int(G))*$\setminus$G contains no any nonempty $\star$-closed set. □

Theorem 3.7. *Let $(X, \tau, \mathcal{I})$ be an ideal topological space and $G \subseteq X$. If G is a weakly $\mathcal{I}_g$-$\star$-closed set, then $cl^*(int(G))\setminus G$ contains no any nonempty $\star$-closed set.*

Proof. Suppose that H is a $\star$-closed set such that H $\subseteq$ cl*(int(G))$\setminus$G. By Theorem 3.6, it follows from the fact that cl*(int(G))$\setminus$G = ((int(G))* $\cup$ int(G))$\setminus$G. □

Theorem 3.8. *Let $(X, \tau, \mathcal{I})$ be an ideal topological space. The following properties are equivalent:*

(1). G is $pre^*_{\mathcal{I}}$-closed for each weakly $\mathcal{I}_g$-$\star$-closed set G in $(X, \tau, \mathcal{I})$

(2). Each singleton $\{x\}$ of X is a $\star$-closed set or $\{x\}$ is $pre^*_{\mathcal{I}}$-open.

Proof. (1) $\Rightarrow$ (2) : Let G be $pre^*_{\mathcal{I}}$-closed for each weakly $\mathcal{I}_g$-$\star$-closed set G in $(X, \tau, \mathcal{I})$ and x $\in$ X. We have cl*(int(G)) $\subseteq$ G for each weakly $\mathcal{I}_g$-$\star$-closed set G in $(X, \tau, \mathcal{I})$. Assume that {x} is not a $\star$-closed set. It follows that X is the only $\star$-open set containing X$\setminus${x}. Then, X$\setminus${x} is a weakly $\mathcal{I}_g$-$\star$-closed set in $(X, \tau, \mathcal{I})$. Thus, cl*(int(X$\setminus${x})) $\subseteq$ X$\setminus${x} and hence {x} $\subseteq$ int*(cl({x})). Consequently, {x} is $pre^*_{\mathcal{I}}$-open.
(2) $\Rightarrow$ (1) : Let G be a weakly $\mathcal{I}_g$-$\star$-closed set in $(X, \tau, \mathcal{I})$. Let x $\in$ cl*(int(G)).
Suppose that {x} is $pre^*_{\mathcal{I}}$-open. We have {x} $\subseteq$ int*(cl({x})). Since x $\in$ cl*(int(G)), then int*(cl({x})) $\cap$ int(G) $\neq \emptyset$. It follows that cl({x}) $\cap$ int(G) $\neq \emptyset$. We have cl({x} $\cap$ int(G)) $\neq \emptyset$ and then {x} $\cap$ int(G) $\neq \emptyset$. Hence, x $\in$ int(G). Thus, we have x $\in$ G. □

Theorem 3.9. *Let $(X, \tau, \mathcal{I})$ be an ideal topological space and $G \subseteq X$. If $cl^*(int(G))\setminus G$ contains no any nonempty $\star$-closed set, then G is a weakly $\mathcal{I}_g$-$\star$-closed set.*

Proof. Suppose that cl*(int(G))$\setminus$G contains no any nonempty $\star$-closed set in $(X, \tau, \mathcal{I})$. Let G $\subseteq$ H and H be an $\star$-open set. Assume that cl*(int(G)) is not contained in H. It follows that cl*(int(G))$\cap$(X$\setminus$H) is a nonempty $\star$-closed subset of cl*(int(G))$\setminus$G. This is a contradiction. Hence G is a weakly $\mathcal{I}_g$-$\star$-closed set. □

Theorem 3.10. *Let $(X, \tau, \mathcal{I})$ be an ideal topological space and $G \subseteq X$. If G is a weakly $\mathcal{I}_g$-$\star$-closed set, then $int(G) = H\setminus K$ where H is $\mathcal{I}$-R closed and K contains no any nonempty $\star$-closed set.*

Proof. Let G be a weakly $\mathcal{I}_g$-$\star$-closed set in $(X, \tau, \mathcal{I})$. Take K = (int(G))*$\setminus$G. Then, by Theorem 3.6, K contains no any nonempty $\star$-closed set. Take H = cl*(int(G)). Then H = cl*(int(H)). Moreover, we have H$\setminus$K = ((int(G))* $\cup$ int(G))$\setminus$((int(G))*$\setminus$G) = ((int(G))* $\cup$ int(G)) $\cap$ (X$\setminus$(int(G))* $\cup$ G) = int(G). □

Theorem 3.11. *Let (X, τ, $\mathcal{I}$) be an ideal topological space and G $\subseteq$ X. Assume that G is a weakly $\mathcal{I}_g$-$\star$-closed set. The following properties are equivalent:*

(1). G is $pre^*_{\mathcal{I}}$-closed,

(2). cl(int(G))\G is a $\star$-closed set,*

(3). (int(G)) \G is a $\star$-closed set.*

Proof. (1) $\Rightarrow$ (2) : Let G be $pre^*_{\mathcal{I}}$-closed. We have cl*(int(G)) $\subseteq$ G. Then, cl*(int(G))\G = $\emptyset$. Thus, cl*(int(G))\G is a $\star$-closed set.

(2) $\Rightarrow$ (1) : Let cl*(int(G))\G be a $\star$-closed set. Since G is a weakly $\mathcal{I}_g$-$\star$-closed set in (X, τ, $\mathcal{I}$), then by Theorem 3.7, cl*(int(G))\G = $\emptyset$. Hence, we have cl*(int(G)) $\subseteq$ G. Thus, G is $pre^*_{\mathcal{I}}$-closed.

(2) $\Leftrightarrow$ (3) : It follows easily from that cl*(int(G))\G = (int(G))*\G. □

Theorem 3.12. *Let (X, τ, $\mathcal{I}$) be an ideal topological space and G $\subseteq$ X be a weakly $\mathcal{I}_g$-$\star$-closed set. Then G $\cup$ (X\(int(G))*) is a weakly $\mathcal{I}_g$-$\star$-closed set in (X, τ, $\mathcal{I}$).*

Proof. Let G be a weakly $\mathcal{I}_g$-$\star$-closed set in (X, τ, $\mathcal{I}$). Suppose that H is an $\star$-open set such that G $\cup$ (X\(int(G))*) $\subseteq$ H. We have X\H $\subseteq$ X\(G $\cup$ (X\(int(G))*)) = (X\G) $\cap$ (int(G))* = (int(G))*\G. Since X\H is a $\star$-closed set and G is a weakly $\mathcal{I}_g$-$\star$-closed set, it follows from Theorem 3.6 that X\H = $\emptyset$. Hence, X = H. Thus, X is the only $\star$-open set containing G $\cup$ (X\(int(G))*). Consequently, G $\cup$ (X\(int(G))*) is a weakly $\mathcal{I}_g$-$\star$-closed set in (X, τ, $\mathcal{I}$). □

Corollary 3.13. *Let (X, τ, $\mathcal{I}$) be an ideal topological space and G $\subseteq$ X be a weakly $\mathcal{I}_g$-$\star$-closed set. Then (int(G))*\G is a weakly $\mathcal{I}_g$-$\star$-open set in (X, τ, $\mathcal{I}$).*

Proof. Since X\((int(G))*\G) = G $\cup$ (X\(int(G))*), it follows from Theorem 3.12 that (int(G))*\G is a weakly $\mathcal{I}_g$-$\star$-open set in (X, τ, $\mathcal{I}$). □

Theorem 3.14. *Let (X, τ, $\mathcal{I}$) be an ideal topological space and G $\subseteq$ X. The following properties are equivalent:*

(1). G is a $\star$-closed and open set,

(2). G is $\mathcal{I}$-R closed and open set,

(3). G is a weakly $\mathcal{I}_g$-$\star$-closed and open set.

Proof. (1) $\Rightarrow$ (2) $\Rightarrow$ (3) : Obvious.

(3) $\Rightarrow$ (1) : Since G is open and weakly $\mathcal{I}_g$-$\star$-closed, cl*(int(G)) $\subseteq$ G and so G = cl*(int(G)). Then G is $\mathcal{I}$-R closed and hence it is $\star$-closed. □

Proposition 3.15. *Every $pre^*_{\mathcal{I}}$-closed set is weakly $\mathcal{I}_g$-$\star$-closed.*

Proof. Let G $\subseteq$ H and H an $\star$-open set in X. Since G is $pre^*_{\mathcal{I}}$-closed, cl*(int(G)) $\subseteq$ G $\subseteq$ H. Hence G is a weakly $\mathcal{I}_g$-$\star$-closed set. □

Example 3.16. *Let (X, τ, $\mathcal{I}$) be an ideal topological space such that X = {a, b, c, d}, τ = {$\emptyset$, {a}, {c}, {a, c}, {c, d}, {a, c, d}, {b, c, d}, X} and $\mathcal{I}$ = {$\emptyset$}. Then {b, c} is a weakly $\mathcal{I}_g$-$\star$-closed set but not $pre^*_{\mathcal{I}}$-closed.*

4. Further Properties

Theorem 4.1. *Let (X, τ, $\mathcal{I}$) be an ideal topological space. The following properties are equivalent:*

(1). Each subset of (X, τ, $\mathcal{I}$) is a weakly $\mathcal{I}_g$-$\star$-closed set,

(2). G is $pre^*_{\mathcal{I}}$-closed for each $\star$-open set G in X.

Proof. (1) $\Rightarrow$ (2) : Suppose that each subset of (X, τ, $\mathcal{I}$) is a weakly $\mathcal{I}_g$-$\star$-closed set. Let G be an $\star$-open set. Since G is weakly $\mathcal{I}_g$-$\star$-closed, then we have cl*(int(G)) $\subseteq$ G. Thus, G is $pre^*_{\mathcal{I}}$-closed.

(2) $\Rightarrow$ (1) : Let H be a subset of (X, τ, $\mathcal{I}$) and G be an $\star$-open set such that H $\subseteq$ G. By (2), we have cl*(int(H)) $\subseteq$ cl*(int(G)) $\subseteq$ G. Thus, H is a weakly $\mathcal{I}_g$-$\star$-closed set in (X, τ, $\mathcal{I}$). □

Theorem 4.2. *Let (X, τ, $\mathcal{I}$) be an ideal topological space. If G is a weakly $\mathcal{I}_g$-$\star$-closed set and G $\subseteq$ H $\subseteq$ cl*(int(G)), then H is a weakly $\mathcal{I}_g$-$\star$-closed set.*

Proof. Let H $\subseteq$ K and K be an $\star$-open set in X. Since G $\subseteq$ K and G is a weakly $\mathcal{I}_g$-$\star$-closed set, then cl*(int(G)) $\subseteq$ K. Since H $\subseteq$ cl*(int(G)), then cl*(int(H)) $\subseteq$ cl*(int(G)) $\subseteq$ K. Thus, cl*(int(H)) $\subseteq$ K and hence, H is a weakly $\mathcal{I}_g$-$\star$-closed set. □

Corollary 4.3. *Let (X, τ, $\mathcal{I}$) be an ideal topological space. If G is a weakly $\mathcal{I}_g$-$\star$-closed and open set, then cl*(G) is a weakly $\mathcal{I}_g$-$\star$-closed set.*

Proof. Let G be a weakly $\mathcal{I}_g$-$\star$-closed and open set in (X, τ, $\mathcal{I}$). We have G $\subseteq$ cl*(G) $\subseteq$ cl*(G) = cl*(int(G)). Hence, by Theorem 4.2, cl*(G) is a weakly $\mathcal{I}_g$-$\star$-closed set in (X, τ, $\mathcal{I}$). □

Theorem 4.4. *Let (X, τ, $\mathcal{I}$) be an ideal topological space and G $\subseteq$ X. If G is a nowhere dense set, then G is a weakly $\mathcal{I}_g$-$\star$-closed set.*

Proof. Let G be a nowhere dense set in X. Since int(G) $\subseteq$ int(cl(G)), then int(G) = $\emptyset$. Hence, cl*(int(G)) = $\emptyset$. Thus, G is a weakly $\mathcal{I}_g$-$\star$-closed set in (X, τ, $\mathcal{I}$). □

Remark 4.5. *The reverse of Theorem 4.4 is not true in general as shown in the following example.*

Example 4.6. *Let X, τ and $\mathcal{I}$ be as in Example 3.16. Then {a, b} is a weakly $\mathcal{I}_g$-$\star$-closed set but not a nowhere dense set.*

Remark 4.7. *(1). The union of two weakly $\mathcal{I}_g$-$\star$-closed sets in an ideal topological space need not be a weakly $\mathcal{I}_g$-$\star$-closed set.*

(2). The intersection of two weakly $\mathcal{I}_g$-$\star$-closed sets in an ideal topological space need not be a weakly $\mathcal{I}_g$-$\star$-closed set.

Example 4.8. *Let (X, τ, $\mathcal{I}$) be an ideal topological space such that X = {a, b, c, d}, τ = {$\emptyset$, {a}, {b}, {a, b}, {a, c}, {a, b, c}, X} and $\mathcal{I}$ = {$\emptyset$, {a}}. Then {a} and {c} are weakly $\mathcal{I}_g$-$\star$-closed sets but their union {a, c} is not a weakly $\mathcal{I}_g$-$\star$-closed set.*

Example 4.9. *Let (X, τ, $\mathcal{I}$) be an ideal topological space such that X = {a, b, c}, τ = {$\emptyset$, {a}, X} and $\mathcal{I}$ = {$\emptyset$}. Then {a, b} and {a, c} are weakly $\mathcal{I}_g$-$\star$-closed sets but their intersection {a} is not a weakly $\mathcal{I}_g$-$\star$-closed set.*

Theorem 4.10. *Let (X, τ, $\mathcal{I}$) be an ideal topological space and G $\subseteq$ X. Then G is a weakly $\mathcal{I}_g$-$\star$-open set if and only if H $\subseteq$ int*(cl(G)) whenever H $\subseteq$ G and H is a $\star$-closed set.*

Proof. Let H be a $\star$-closed set in X and H $\subseteq$ G. It follows that X\H is an $\star$-open set and X\G $\subseteq$ X\H. Since X\G is a weakly $\mathcal{I}_g$-$\star$-closed set, then cl*(int(X\G)) $\subseteq$ X\H. We have X\int*(cl(G)) $\subseteq$ X\H. Thus, H $\subseteq$ int*(cl(G)). □

Theorem 4.11. *Let (X, τ, $\mathcal{I}$) be an ideal topological space and G $\subseteq$ X. If G is a weakly $\mathcal{I}_g$-$\star$-closed set, then cl*(int(G))\G is a weakly $\mathcal{I}_g$-$\star$-open set in (X, τ, $\mathcal{I}$).*

Proof. Let G be a weakly $\mathcal{I}_g$-$\star$-closed set in (X, τ, $\mathcal{I}$). Suppose that H is a $\star$-closed set such that H $\subseteq$ cl*(int(G))\G. Since G is a weakly $\mathcal{I}_g$-$\star$-closed set, it follows from Theorem 3.7 that H = $\emptyset$. Thus, we have H $\subseteq$ int*(cl(cl*(int(G))\G)). It follows from Theorem 4.10 that cl*(int(G))\G is a weakly $\mathcal{I}_g$-$\star$-open set in (X, τ, $\mathcal{I}$). □

Theorem 4.12. *Let (X, τ, $\mathcal{I}$) be an ideal topological space and G $\subseteq$ X. If G is a weakly $\mathcal{I}_g$-$\star$-open set, then H = X whenever H is an $\star$-open set and int*(cl(G)) $\cup$ (X\G) $\subseteq$ H.*

Proof. Let H be an $\star$-open set in X and int*(cl(G)) $\cup$ (X\G) $\subseteq$ H. We have X\H $\subseteq$ (X\int*(cl(G))) $\cap$ G = cl*(int(X\G))\(X\G). Since X\H is a $\star$-closed set and X\G is a weakly $\mathcal{I}_g$-$\star$-closed set, it follows from Theorem 3.7 that X\H = $\emptyset$. Thus, we have H = X. □

Theorem 4.13. *Let (X, τ, $\mathcal{I}$) be an ideal topological space. If G is a weakly $\mathcal{I}_g$-$\star$-open set and int*(cl(G)) $\subseteq$ H $\subseteq$ G, then H is a weakly $\mathcal{I}_g$-$\star$-open set.*

Proof. Let G be a weakly $\mathcal{I}_g$-$\star$-open set and int*(cl(G)) $\subseteq$ H $\subseteq$ G. Since int*(cl(G)) $\subseteq$ H $\subseteq$ G, then int*(cl(G)) = int*(cl(H)). Let K be a $\star$-closed set and K $\subseteq$ H. We have K $\subseteq$ G. Since G is a weakly $\mathcal{I}_g$-$\star$-open set, it follows from Theorem 4.10 that K $\subseteq$ int*(cl(G)) = int*(cl(H)). Hence, by Theorem 4.10, H is a weakly $\mathcal{I}_g$-$\star$-open set in (X, τ, $\mathcal{I}$). □

Corollary 4.14. *Let (X, τ, $\mathcal{I}$) be an ideal topological space and G $\subseteq$ X. If G is a weakly $\mathcal{I}_g$-$\star$-open and $\star$-closed set, then int*(G) is a weakly $\mathcal{I}_g$-$\star$-open set.*

Proof. Let G be a weakly $\mathcal{I}_g$-$\star$-open and $\star$-closed set in (X, τ, $\mathcal{I}$). Then int*(cl(G)) = int*(G) $\subseteq$ int*(G) $\subseteq$ G. Thus, by Theorem 4.13, int*(G) is a weakly $\mathcal{I}_g$-$\star$-open set in (X, τ, $\mathcal{I}$). □

Definition 4.15. *A subset J of an ideal topological space (X, τ, $\mathcal{I}$) is called an $O_{\mathcal{I}}$-set if J = M$\cup$N where M is $\star$-closed and N is $pre^*_{\mathcal{I}}$-open.*

Remark 4.16. *Every $pre^*_{\mathcal{I}}$-open (resp. $\star$-closed) set is an $O_{\mathcal{I}}$-set but not conversely.*

Example 4.17. *Let X, τ and $\mathcal{I}$ be as in Example 3.16. Then $\{b\}$ is an $O_{\mathcal{I}}$-set but not $pre^*_{\mathcal{I}}$-open. Also $\{c\}$ is an $O_{\mathcal{I}}$-set but not $\star$-closed.*

Theorem 4.18. *For a subset H of (X, τ, $\mathcal{I}$), the following are equivalent.*

(1). H is $pre^*_{\mathcal{I}}$-open.

(2). H is an $O_{\mathcal{I}}$-set and weakly $\mathcal{I}_g$-$\star$-open.

Proof. (1) ⇒ (2): By Remark 4.16, H is an $O_{\mathcal{I}}$-set. By Proposition 3.15, H is weakly $\mathcal{I}_g$-$\star$-open.
(2) ⇒ (1): Let H be an $O_{\mathcal{I}}$-set and weakly $\mathcal{I}_g$-$\star$-open. Then there exist a $\star$-closed set M and a $pre^*_{\mathcal{I}}$-open set N such that H =M$\cup$N. Since M $\subseteq$ H and H is weakly $\mathcal{I}_g$-$\star$-open, by Theorem 4.10, M $\subseteq$ int*(cl(H)). Also, we have N $\subseteq$ int*(cl(N)). Since N $\subseteq$ H, N $\subseteq$ int*(cl(N)) $\subseteq$ int*(cl(H)). Then H = M$\cup$N $\subseteq$ int*(cl(H)). So H is $pre^*_{\mathcal{I}}$-open. □

The following Example shows that the concepts of weakly $\mathcal{I}_g$-$\star$-open set and $O_{\mathcal{I}}$-set are independent.

Example 4.19. *Let X, τ and $\mathcal{I}$ be as in Example 3.16. Then $\{d\}$ is a weakly $\mathcal{I}_g$-$\star$-open set but not an $O_{\mathcal{I}}$-set. Also $\{b\}$ is an $O_{\mathcal{I}}$-set but not weakly $\mathcal{I}_g$-$\star$-open.*

5. $\star$-$pre^*_{\mathcal{I}}$-normal Spaces

Definition 5.1. *An ideal topological space (X, τ, $\mathcal{I}$) is said to be $\star$-$pre^*_{\mathcal{I}}$-normal if for every pair of disjoint $\star$-closed subsets A, B of X, there exist disjoint $pre^*_{\mathcal{I}}$-open sets U, V of X such that A ⊆ U and B ⊆ V.*

Theorem 5.2. *The following properties are equivalent for a space (X, τ, $\mathcal{I}$).*

(1). X is $\star$-$pre^*_{\mathcal{I}}$-normal;

(2). for any disjoint $\star$-closed sets A and B, there exist disjoint weakly $\mathcal{I}_g$-$\star$-open sets U, V of X such that A ⊆ U and B ⊆ V;

(3). for any $\star$-closed set A and any $\star$-open set B containing A, there exists a weakly $\mathcal{I}_g$-$\star$-open set U such that A ⊆ U ⊆ cl(int(U)) ⊆ B.*

Proof. (1) ⇒ (2): The proof is obvious.

(2) ⇒ (3): Let A be any $\star$-closed set of X and B any $\star$-open set of X such that A ⊆ B. Then A and X\B are disjoint $\star$-closed sets of X. By (2), there exist disjoint weakly $\mathcal{I}_g$-$\star$-open sets U, V of X such that A ⊆ U and X\B ⊆ V. Since V is weakly $\mathcal{I}_g$-$\star$-open set, by Theorem 4.10, X\B ⊆ int*(cl(V)) and U∩int*(cl(V)) = ∅. Therefore we obtain cl*(int(U)) ⊆ cl*(int(X\V)) and hence A ⊆ U ⊆ cl*(int(U)) ⊆ B.

(3) ⇒ (1): Let A and B be any disjoint $\star$-closed sets of X. Then A ⊆ X\B and X\B is $\star$-open and hence there exists a weakly $\mathcal{I}_g$-$\star$-open set G of X such that A ⊆ G ⊆ cl*(int(G)) ⊆ X\B. Put U = int*(cl(G)) and V = X\cl*(int(G)). Then U and V are disjoint $pre^*_{\mathcal{I}}$-open sets of X such that A ⊆ U and B ⊆ V. Therefore X is $\star$-$pre^*_{\mathcal{I}}$-normal. □

Definition 5.3. *A function f : (X, τ, $\mathcal{I}$) → (Y, σ) is said to be weakly $\mathcal{I}_g$-$\star$-continuous if $f^{-1}(V)$ is weakly $\mathcal{I}_g$-$\star$-closed in X for every closed set V of Y.*

Definition 5.4. *A function f : (X, τ, $\mathcal{I}$) → (Y, σ, $\mathcal{J}$) is called weakly $(\mathcal{I}, \mathcal{J})_g$-$\star$-irresolute if $f^{-1}(V)$ is weakly $\mathcal{I}_g$-$\star$-closed in X for every weakly $\mathcal{J}_g$-$\star$-closed of Y.*

Definition 5.5. *A function $f : (X, \tau, \mathcal{I}) \to (Y, \sigma)$ is said to be H$\star$-closed if the image of every $\star$-closed set of X is closed in Y.*

Theorem 5.6. *Let $f : (X, \tau, \mathcal{I}) \to (Y, \sigma, \mathcal{J})$ be a weakly $\mathcal{I}_g$-$\star$-continuous H$\star$-closed injection. If Y is $\star$-normal, then X is $\star$-$pre^*_{\mathcal{I}}$-normal.*

Proof. Let A and B be disjoint $\star$-closed sets of X. Since f is H$\star$-closed injection, f(A) and f(B) are disjoint closed sets of Y. By the $\star$-normality of Y, there exist disjoint $\star$-open sets U and V such that f(A) ⊆ U and f(B) ⊆ V. Since f is weakly $\mathcal{I}_g$-$\star$-continuous, then $f^{-1}(U)$ and $f^{-1}(V)$ are disjoint weakly $\mathcal{I}_g$-$\star$-open sets such that A ⊆ $f^{-1}(U)$ and B ⊆ $f^{-1}(V)$. Therefore X is $\star$-$pre^*_{\mathcal{I}}$-normal by Theorem 5.2. □

Theorem 5.7. *Let $f : (X, \tau, \mathcal{I}) \to (Y, \sigma, \mathcal{J})$ be a weakly $(\mathcal{I}, \mathcal{J})_g$-$\star$-irresolute H$\star$-closed injection. If Y is $\star$-$pre^*_{\mathcal{I}}$-normal, then X is $\star$-$pre^*_{\mathcal{I}}$-normal.*

Proof. Let A and B be disjoint $\star$-closed sets of X. Since f is H$\star$-closed injection, f(A) and f(B) are disjoint closed (and hence $\star$-closed) sets of Y. Since Y is $\star$-$pre^*_{\mathcal{I}}$-normal, by Theorem 5.2, there exist disjoint weakly $\mathcal{J}_g$-$\star$-open sets U and V such that f(A) ⊆ U and f(B) ⊆ V. Since f is weakly $(\mathcal{I}, \mathcal{J})_g$-$\star$-irresolute, then $f^{-1}(U)$ and $f^{-1}(V)$ are disjoint weakly $\mathcal{I}_g$-$\star$-open sets of X such that A ⊆ $f^{-1}(U)$ and B ⊆ $f^{-1}(V)$. Therefore X is $\star$-$pre^*_{\mathcal{I}}$-normal. □

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