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$\mathcal{I}_{g^\#}$-Normal and $\mathcal{I}_{g^\#}$-Regular Spaces

Research Article

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Abstract: $\mathcal{I}_{g^\#}$-normal and $\mathcal{I}_{g^\#}$-regular spaces are introduced and various characterizations and properties are given. Characterizations of normal, mildly normal, $g^\#$-normal and regular spaces are also given.

MSC: 54D10, 54D15.

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1. Introduction and Preliminaries

By a space, we always mean a topological space (X, τ) with no separation properties assumed. If A⊆X, cl(A) and int(A) will, respectively, denote the closure and interior of A in (X, τ). A subset A of a space (X, τ) is said to be regular open [17] if A=int(cl(A)) and A is said to be regular closed [17] if A=cl(int(A)). A subset A of a space (X, τ) is said to be an α-open [12] (resp. preopen [9]) if A⊆int(cl(int(A))) (resp. A⊆int(cl(A))).

The complement of α-open set is α-closed [10]. The α-closure [10] of a subset A of X, denoted by αcl(A), is defined to be the intersection of all α-closed sets containing A. The α-interior [10] of a subset A of X, denoted by αint(A), is defined to be the union of all α-open sets contained in A. The family of all α-open sets in (X, τ), denoted by τ^α, is a topology on X finer than τ. The interior of a subset A in (X, τ^α) is denoted by int_α(A). The closure of a subset A in (X, τ^α) is denoted by cl_α(A). A subset A of a space (X, τ) is said to be αg-closed [8] if $\mathrm{cl}_\alpha(A) \subseteq U$ whenever $A \subseteq U$ and U is open.

The complement of αg-closed set is αg-open. A subset A of a space (X, τ) is said to be $g^\#$-closed [19] if cl(A)⊆U whenever A⊆U and U is αg-open. The complement of $g^\#$-closed set is $g^\#$-open. A subset A of a space (X, τ) is said to be $\alpha g^\#$-closed [5] (resp. rαg-closed [13]) if cl_α(A)⊆U whenever A⊆U and U is αg-open (resp. regular open). A is said to be $\alpha g^\#$-open (resp. rαg-open) if X−A is $\alpha g^\#$-closed (resp. rαg-closed). A subset A of a space (X, τ) is said to be g-closed [7] if cl(A)⊆U whenever A⊆U and U is open. A space (X, τ) is said to be $g^\#$-normal [5], if for every disjoint $g^\#$-closed sets A and B, there exist disjoint open sets U and V such that A⊆U and B⊆V.

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An ideal $\mathcal{I}$ on a topological space (X, τ) is a nonempty collection of subsets of X which satisfies (i) A$\in\mathcal{I}$ and B$\subseteq$A imply B$\in\mathcal{I}$ and (ii) A$\in\mathcal{I}$ and B$\in\mathcal{I}$ imply A$\cup$B$\in\mathcal{I}$ [6]. Given a topological space (X, τ) with an ideal $\mathcal{I}$ on X and if $\wp$(X) is the set of all subsets of X, a set operator (.)* : $\wp$(X) $\rightarrow$ $\wp$(X), called a local function [6] of A with respect to τ and $\mathcal{I}$ is defined as follows: for A$\subseteq$X, A*($\mathcal{I}$, τ)={x$\in$X : U$\cap$A$\notin\mathcal{I}$ for every U$\in\tau$(x)} where τ(x)={U$\in\tau$: x$\in$U}.

We will make use of the basic facts about the local functions [[4], Theorem 2.3] without mentioning it explicitly. A Kuratowski closure operator cl*(.) for a topology $\tau^*(\mathcal{I}, \tau)$, called the $\star$-topology, finer than τ is defined by cl*(A)=A$\cup$A*($\mathcal{I}$, τ) [18]. When there is no chance for confusion, we will simply write A* for A*($\mathcal{I}$, τ) and τ^* for $\tau^*(\mathcal{I}, \tau)$. int*(A) will denote the interior of A in (X, τ^*). If $\mathcal{I}$ is an ideal on X, then (X, τ, $\mathcal{I}$) is called an ideal topological space. $\mathcal{N}$ is the ideal of all nowhere dense subsets in (X, τ).

A subset A of an ideal topological space (X, τ, $\mathcal{I}$) is τ^*-closed [4] or $\star$-closed (resp. $*$-dense in itself [3]) if A*$\subseteq$A (resp. A$\subseteq$A*). A subset A of an ideal topological space (X, τ, $\mathcal{I}$) is $\mathcal{I}_{g\#}$-closed [5] if A*$\subseteq$U whenever U is αg-open and A$\subseteq$U. By Theorem 2.5 of [5], every $\star$-closed and hence every closed set is $\mathcal{I}_{g\#}$-closed. A subset A of an ideal topological space (X, τ, $\mathcal{I}$) is said to be $\mathcal{I}_{g\#}$-open [5] if X$-$A is $\mathcal{I}_{g\#}$-closed.

In this paper, we define $\mathcal{I}_{g\#}$-normal, $_{g\#}\mathcal{I}$-normal and $\mathcal{I}_{g\#}$-regular spaces using $\mathcal{I}_{g\#}$-open sets and give characterizations and properties of such spaces. Also, characterizations of normal, mildly normal, $g^\#$-normal and regular spaces are given.

An ideal $\mathcal{I}$ is said to be codense [2] if $\tau\cap\mathcal{I}$={ϕ}. $\mathcal{I}$ is said to be completely codense [15] if PO(X)$\cap\mathcal{I}$={ϕ}, where PO(X) is the family of all preopen sets in (X, τ). Every completely codense ideal is codense but not conversely [15]. The following lemmas and proposition will be useful in the sequel.

Lemma 1.1 ([15], Theorem 6)**.** *Let (X, τ, $\mathcal{I}$) be an ideal topological space. If $\mathcal{I}$ is completely codense, then $\tau^*\subseteq\tau^\alpha$.*

Lemma 1.2 ([5], Theorem 2.26)**.** *Let (X, τ, $\mathcal{I}$) be an ideal topological space where $\mathcal{I}$ is completely codense. Then the following are equivalent.*

(1). X is normal.

(2). For any disjoint closed sets A and B, there exist disjoint $\mathcal{I}_{g\#}$-open sets U and V such that A$\subseteq$U and B$\subseteq$V.

(3). For any closed set A and open set V containing A, there exists an $\mathcal{I}_{g\#}$-open set U such that A$\subseteq$U$\subseteq$cl*(U)$\subseteq$V.

Lemma 1.3 ([5])**.** *If (X, τ, $\mathcal{I}$) is an ideal topological space and A$\subseteq$X, then the following hold.*

(1). If $\mathcal{I}$={ϕ}, then A is $\mathcal{I}_{g\#}$-closed if and only if A is $g^\#$-closed.

(2). If $\mathcal{I}=\mathcal{N}$, then A is $\mathcal{I}_{g\#}$-closed if and only if A is $\alpha g^\#$-closed.

Lemma 1.4 ([5], Theorem 2.4)**.** *If (X, τ, $\mathcal{I}$) is an ideal topological space and A$\subseteq$X, then the following are equivalent.*

(1). A is $\mathcal{I}_{g\#}$-closed.

(2). cl*(A)$\subseteq$U whenever A$\subseteq$U and U is αg-open in X.

Lemma 1.5 ([5], Theorem 2.25)**.** *Let (X, τ, $\mathcal{I}$) be an ideal topological space and A$\subseteq$X. Then A is $\mathcal{I}_{g\#}$-open if and only if F$\subseteq$int*(A) whenever F is αg-closed and F$\subseteq$A.*

Lemma 1.6 ([5], Theorem 2.24)**.** *Let (X, τ, $\mathcal{I}$) be an ideal topological space. Then every subset of X is $\mathcal{I}_{g\#}$-closed if and only if every αg-open set is $\star$-closed.*

Proposition 1.7 ([8])**.** *In a space X, the following hold:*

(1). Every open set is αg-open but not conversely.

(2). Every $\alpha g^{\#}$-open set is αg-open but not conversely.

2. $\mathcal{I}_{g\#}$-normal and ${}_{g\#}\mathcal{I}$-normal spaces

An ideal topological space (X, τ, $\mathcal{I}$) is said to be an $\mathcal{I}_{g\#}$-normal space if for every pair of disjoint closed sets A and B, there exist disjoint $\mathcal{I}_{g\#}$-open sets U and V such that A$\subseteq$U and B$\subseteq$V. Since every open set is an $\mathcal{I}_{g\#}$-open set, every normal space is $\mathcal{I}_{g\#}$-normal. The following Example 2.1 shows that an $\mathcal{I}_{g\#}$-normal space is not necessarily a normal space. Theorem 2.2 below gives characterizations of $\mathcal{I}_{g\#}$-normal spaces. Theorem 2.3 below shows that the two concepts coincide for completely codense ideal topological spaces.

Example 2.1. *Let X=$\{a, b, c\}$, τ=$\{\phi, \{b\}, \{a, b\}, \{b, c\}, X\}$ and $\mathcal{I}$=$\{\phi, \{b\}\}$. Then ϕ^*=ϕ, $(\{a, b\})^*$=$\{a\}$, $(\{b, c\})^*$=$\{c\}$, $(\{b\})^*$=ϕ and X^*=$\{a, c\}$. Here every αg-open set is $\star$-closed and so, by Lemma 1.6, every subset of X is $\mathcal{I}_{g\#}$-closed and hence every subset of X is $\mathcal{I}_{g\#}$-open. This implies that (X, τ, $\mathcal{I}$) is $\mathcal{I}_{g\#}$-normal. Now $\{a\}$ and $\{c\}$ are disjoint closed subsets of X which are not separated by disjoint open sets and so (X, τ) is not normal.*

Theorem 2.2. *Let (X, τ, $\mathcal{I}$) be an ideal topological space. Then the following are equivalent.*

(1). X is $\mathcal{I}_{g\#}$-normal.

(2). For every closed set A and an open set V containing A, there exists an $\mathcal{I}_{g\#}$-open set U such that A$\subseteq$U$\subseteq$cl*(U)$\subseteq$V.

Proof.
(1)$\Rightarrow$(2). Let A be a closed set and V be an open set containing A. Since A and X$-$V are disjoint closed sets, there exist disjoint $\mathcal{I}_{g\#}$-open sets U and W such that A$\subseteq$U and X$-$V$\subseteq$W. Again, U$\cap$W=ϕ implies that U$\cap$int*(W)=ϕ and so cl*(U)$\subseteq$X$-$int*(W). Since X$-$V is αg-closed and W is $\mathcal{I}_{g\#}$-open, X$-$V$\subseteq$W implies that X$-$V$\subseteq$int*(W) and so X$-$int*(W)$\subseteq$V. Thus, we have A$\subseteq$U$\subseteq$cl*(U)$\subseteq$X$-$int*(W)$\subseteq$V which proves (2).

(2)$\Rightarrow$(1). Let A and B be two disjoint closed subsets of X. By hypothesis, there exists an $\mathcal{I}_{g\#}$-open set U such that A$\subseteq$U$\subseteq$cl*(U)$\subseteq$X$-$B. If W=X$-$cl*(U), then U and W are the required disjoint $\mathcal{I}_{g\#}$-open sets containing A and B respectively. So, (X, τ, $\mathcal{I}$) is $\mathcal{I}_{g\#}$-normal. □

Theorem 2.3. *Let (X, τ, $\mathcal{I}$) be an ideal topological space where $\mathcal{I}$ is completely codense. If (X, τ, $\mathcal{I}$) is $\mathcal{I}_{g\#}$-normal, then it is a normal space.*

Proof. It is obvious from Theorem 2.2 and Lemma 1.2. □

Theorem 2.4. *Let (X, τ, $\mathcal{I}$) be an $\mathcal{I}_{g\#}$-normal space. If F is closed and A is a $g^{\#}$-closed set such that A$\cap$F=ϕ, then there exist disjoint $\mathcal{I}_{g\#}$-open sets U and V such that A$\subseteq$U and F$\subseteq$V.*

Proof. Since A$\cap$F=ϕ, A$\subseteq$X$-$F where X$-$F is αg-open. Therefore, by hypothesis, cl(A)$\subseteq$X$-$F. Since cl(A)$\cap$F=ϕ and X is $\mathcal{I}_{g\#}$-normal, there exist disjoint $\mathcal{I}_{g\#}$-open sets U and V such that cl(A)$\subseteq$U and F$\subseteq$V. Thus A $\subseteq$ U and F $\subseteq$ V. □

The following Corollaries 2.5 and 2.6 give properties of normal spaces. If $\mathcal{I}=\{\phi\}$ in Theorem 2.4, then we have the following Corollary 2.5, the proof of which follows from Theorem 2.3 and Lemma 1.3, since $\{\phi\}$ is a completely codense ideal. If $\mathcal{I}=\mathcal{N}$ in Theorem 2.4, then we have the Corollary 2.6 below, since $\tau^*(\mathcal{N})=\tau^\alpha$ and $\mathcal{I}_{g\#}$-open sets coincide with $\alpha g^{\#}$-open sets.

Corollary 2.5. *Let (X, τ) be a normal space with $\mathcal{I} = \{\phi\}$. If F is a closed set and A is a $g^{\#}$-closed set disjoint from F, then there exist disjoint $g^{\#}$-open sets U and V such that A⊆U and F⊆V.*

Corollary 2.6. *Let (X, τ, $\mathcal{I}$) be a normal ideal topological space where $\mathcal{I}=\mathcal{N}$. If F is a closed set and A is a $g^{\#}$-closed set disjoint from F, then there exist disjoint $\alpha g^{\#}$-open sets U and V such that A⊆U and F⊆V.*

Theorem 2.7. *Let (X, τ, $\mathcal{I}$) be an ideal topological space which is $\mathcal{I}_{g\#}$-normal. Then the following hold.*

(1). For every closed set A and every $g^{\#}$-open set B containing A, there exists an $\mathcal{I}_{g\#}$-open set U such that A⊆int*(U)⊆U⊆B.

(2). For every $g^{\#}$-closed set A and every open set B containing A, there exists an $\mathcal{I}_{g\#}$-closed set U such that A⊆U⊆cl*(U)⊆B.

Proof.

(1). Let A be a closed set and B be a $g^{\#}$-open set containing A. Then A∩(X−B)=ϕ, where A is closed and X−B is $g^{\#}$-closed. By Theorem 2.4, there exist disjoint $\mathcal{I}_{g\#}$-open sets U and V such that A⊆U and X−B⊆V. Since U∩V=ϕ, we have U⊆X−V. By Lemma 1.5, A⊆int*(U). Therefore, A⊆int*(U)⊆U⊆X−V⊆B. This proves (1).

(2). Let A be a $g^{\#}$-closed set and B be an open set containing A. Then X−B is a closed set contained in the $g^{\#}$-open set X−A. By (1), there exists an $\mathcal{I}_{g\#}$-open set V such that X−B⊆int*(V)⊆V⊆X−A. Therefore, A⊆X−V⊆cl*(X−V)⊆B. If U=X−V, then A⊆U⊆cl*(U)⊆B and so U is the required $\mathcal{I}_{g\#}$-closed set.

□

The following Corollaries 2.8 and 2.9 give some properties of normal spaces. If $\mathcal{I}=\{\phi\}$ in Theorem 2.7, then we have the following Corollary 2.8. If $\mathcal{I}=\mathcal{N}$ in Theorem 2.7, then we have the Corollary 2.9 below.

Corollary 2.8. *Let (X, τ) be a normal space with $\mathcal{I}=\{\phi\}$. Then the following hold.*

(1). For every closed set A and every $g^{\#}$-open set B containing A, there exists a $g^{\#}$-open set U such that A⊆int(U)⊆U⊆B.

(2). For every $g^{\#}$-closed set A and every open set B containing A, there exists a $g^{\#}$-closed set U such that A⊆U⊆cl(U)⊆B.

Corollary 2.9. *Let (X, τ) be a normal space with $\mathcal{I} = \mathcal{N}$. Then the following hold.*

(1). For every closed set A and every $g^{\#}$-open set B containing A, there exists an $\alpha g^{\#}$-open set U such that A⊆int_α(U)⊆U⊆B.

(2). For every $g^{\#}$-closed set A and every open set B containing A, there exists an $\alpha g^{\#}$-closed set U such that A⊆U⊆cl_α(U)⊆B.

An ideal topological space (X, τ, $\mathcal{I}$) is said to be $_{g\#}\mathcal{I}$-normal if for each pair of disjoint $\mathcal{I}_{g\#}$-closed sets A and B, there exist disjoint open sets U and V in X such that A⊆U and B⊆V. Since every closed set is $\mathcal{I}_{g\#}$-closed, every $_{g\#}\mathcal{I}$-normal space is normal. But a normal space need not be $_{g\#}\mathcal{I}$-normal as the following Example 2.10 shows. Theorems 2.11 and 2.13 below give characterizations of $_{g\#}\mathcal{I}$-normal spaces.

Example 2.10. *Let X={a, b, c}, τ = {ϕ, X, {a}, {b, c}} and $\mathcal{I}$ = {ϕ, {a}, {b}, {c}, {a, b}, {a, c}, {b, c}, X}. Every αg-open set is $\star$-closed and so every subset of X is $\mathcal{I}_{g\#}$-closed. Now A={a, b} and B={c} are disjoint $\mathcal{I}_{g\#}$-closed sets, but they are not separated by disjoint open sets. So (X, τ, $\mathcal{I}$) is not ${}_{g\#}\mathcal{I}$-normal. But (X, τ, $\mathcal{I}$) is normal.*

Theorem 2.11. *In an ideal topological space (X, τ, $\mathcal{I}$), the following are equivalent.*

(1). X is ${}_{g\#}\mathcal{I}$-normal.

(2). For every $\mathcal{I}_{g\#}$-closed set A and every $\mathcal{I}_{g\#}$-open set B containing A, there exists an open set U of X such that A⊆U⊆cl(U)⊆B.

Proof. It is similar to the proof of Theorem 2.2. □

If $\mathcal{I}$={ϕ}, then ${}_{g\#}\mathcal{I}$-normal spaces coincide with $g^{\#}$-normal spaces and so if we take $\mathcal{I}$={ϕ}, in Theorem 2.11, then we have the following characterization for $g^{\#}$-normal spaces.

Corollary 2.12. *In a space (X, τ), the following are equivalent.*

(1). X is $g^{\#}$-normal.

(2). For every $g^{\#}$-closed set A and every $g^{\#}$-open set B containing A, there exists an open set U of X such that A⊆U⊆cl(U)⊆B.

Theorem 2.13. *In an ideal topological space (X, τ, $\mathcal{I}$), the following are equivalent.*

(1). X is ${}_{g\#}\mathcal{I}$-normal.

(2). For each pair of disjoint $\mathcal{I}_{g\#}$-closed subsets A and B of X, there exists an open set U of X containing A such that cl(U)∩B=ϕ.

(3). For each pair of disjoint $\mathcal{I}_{g\#}$-closed subsets A and B of X, there exist an open set U containing A and an open set V containing B such that cl(U)∩cl(V)=ϕ.

Proof.
(1)⇒(2). Suppose that A and B are disjoint $\mathcal{I}_{g\#}$-closed subsets of X. Then the $\mathcal{I}_{g\#}$-closed set A is contained in the $\mathcal{I}_{g\#}$-open set X−B. By Theorem 2.11, there exists an open set U such that A⊆U⊆cl(U)⊆X−B. Therefore, U is the required open set containing A such that cl(U)∩B=ϕ.

(2)⇒(3). Let A and B be two disjoint $\mathcal{I}_{g\#}$-closed subsets of X. By hypothesis, there exists an open set U of X containing A such that cl(U)∩B=ϕ. Also, cl(U) and B are disjoint $\mathcal{I}_{g\#}$-closed sets of X. By hypothesis, there exists an open set V of X containing B such that cl(U)∩cl(V)=ϕ.

(3)⇒(1). The proof is clear. □

If $\mathcal{I}$={ϕ}, in Theorem 2.13, then we have the following characterizations for $g^{\#}$-normal spaces.

Corollary 2.14. *Let (X, τ) be a space. Then the following are equivalent.*

(1). X is $g^{\#}$-normal.

(2). For each pair of disjoint $g^\#$-closed subsets A and B of X, there exists an open set U of X containing A such that cl(U)∩B=ϕ.

(3). For each pair of disjoint $g^\#$-closed subsets A and B of X, there exist an open set U containing A and an open set V containing B such that cl(U)∩cl(V)=ϕ.

Theorem 2.15. *Let (X, τ, $\mathcal{I}$) be an ${}_{g^\#}\mathcal{I}$-normal space. If A and B are disjoint $\mathcal{I}_{g^\#}$-closed subsets of X, then there exist disjoint open sets U and V such that $cl^\star(A)$⊆U and $cl^\star(B)$⊆V.*

Proof. Suppose that A and B are disjoint $\mathcal{I}_{g^\#}$-closed sets. By Theorem 2.13(3), there exist an open set U containing A and an open set V containing B such that cl(U)∩cl(V)=ϕ. Since A is $\mathcal{I}_{g^\#}$-closed, A⊆U implies that $cl^\star(A)$⊆U. Similarly $cl^\star(B)$⊆V.

If $\mathcal{I}$={ϕ}, in Theorem 2.15, then we have the following property of disjoint $g^\#$-closed sets in $g^\#$-normal spaces. □

Corollary 2.16. *Let (X, τ) be a $g^\#$-normal space. If A and B are disjoint $g^\#$-closed subsets of X, then there exist disjoint open sets U and V such that cl(A)⊆U and cl(B)⊆V.*

Theorem 2.17. *Let (X, τ, $\mathcal{I}$) be an ${}_{g^\#}\mathcal{I}$-normal space. If A is an $\mathcal{I}_{g^\#}$-closed set and B is an $\mathcal{I}_{g^\#}$-open set containing A, then there exists an open set U such that A⊆$cl^\star(A)$⊆U⊆$int^\star(B)$⊆B.*

Proof. Suppose A is an $\mathcal{I}_{g^\#}$-closed set and B is an $\mathcal{I}_{g^\#}$-open set containing A. Since A and X−B are disjoint $\mathcal{I}_{g^\#}$-closed sets, by Theorem 2.15, there exist disjoint open sets U and V such that $cl^\star(A)$⊆U and $cl^\star(X-B)$⊆V. Now, X−$int^\star(B)$=$cl^\star(X-B)$⊆V implies that X−V⊆$int^\star(B)$. Again, U∩V=ϕ implies U⊆X−V and so A⊆$cl^\star(A)$⊆U⊆X−V⊆$int^\star(B)$⊆B. □

If $\mathcal{I}$={ϕ}, in Theorem 2.17, then we have the following Corollary 2.18.

Corollary 2.18. *Let (X, τ) be a $g^\#$-normal space. If A is a $g^\#$-closed set and B is a $g^\#$-open set containing A, then there exists an open set U such that A⊆cl(A)⊆U⊆int(B)⊆B.*

The following Theorem 2.19 gives a characterization of normal spaces in terms of $g^\#$-open sets which follows from Lemma 1.2 if $\mathcal{I}$={ϕ}.

Theorem 2.19. *Let (X, τ) be a space. Then the following are equivalent.*

(1). X is normal.

(2). For any disjoint closed sets A and B, there exist disjoint $g^\#$-open sets U and V such that A⊆U and B⊆V.

(3). For any closed set A and open set V containing A, there exists a $g^\#$-open set U such that A⊆U⊆cl(U)⊆V.

The rest of the section is devoted to the study of mildly normal spaces in terms of $\mathcal{I}_{g^\#}$-open sets, $\mathcal{I}_g$-open sets and $\mathcal{I}_{rg}$-open sets. A space (X, τ) is said to be a mildly normal space [16] if disjoint regular closed sets are separated by disjoint open sets. A subset A of a space (X, τ) is said to be rg-closed [14] if cl(A)⊆U whenever A⊆U and U is regular open in X. The complement of rg-closed set is called rg-open.

A subset A of an ideal topological space (X, τ, $\mathcal{I}$) is said to be $\mathcal{I}_g$-closed [1] if A*⊆U whenever A⊆U and U is open. A subset A of an ideal topological space (X, τ, $\mathcal{I}$) is said to be a regular generalized closed set with respect to an ideal $\mathcal{I}$

($\mathcal{I}_{rg}$-closed) [11] if A*⊆U whenever A⊆U and U is regular open. A is called $\mathcal{I}_g$-open (resp. $\mathcal{I}_{rg}$-open) if X−A is $\mathcal{I}_g$-closed (resp. $\mathcal{I}_{rg}$-closed).

Clearly, every $\mathcal{I}_{g\#}$-closed set is $\mathcal{I}_g$-closed and every $\mathcal{I}_g$-closed set is $\mathcal{I}_{rg}$-closed but the separate converses are not true. Theorem 2.21 below gives characterizations of mildly normal spaces. Corollary 2.22 below gives characterizations of mildly normal spaces in terms of $\alpha g^{\#}$-open, αg-open and rαg-open sets. Corollary 2.23 below gives characterizations of mildly normal spaces in terms of $g^{\#}$-open, g-open and rg-open sets. The following Lemma 2.20 is essential to prove Theorem 2.21.

Lemma 2.20 ([11]). *Let (X, τ, $\mathcal{I}$) be an ideal topological space. A subset A⊆X is $\mathcal{I}_{rg}$-open if and only if F⊆int*(A) whenever F is regular closed and F⊆A.*

Theorem 2.21. *Let (X, τ, $\mathcal{I}$) be an ideal topological space where $\mathcal{I}$ is completely codense. Then the following are equivalent.*

(1). X is mildly normal.

(2). For disjoint regular closed sets A and B, there exist disjoint $\mathcal{I}_{g\#}$-open sets U and V such that A⊆U and B⊆V.

(3). For disjoint regular closed sets A and B, there exist disjoint $\mathcal{I}_g$-open sets U and V such that A⊆U and B⊆V.

(4). For disjoint regular closed sets A and B, there exist disjoint $\mathcal{I}_{rg}$-open sets U and V such that A⊆U and B⊆V.

(5). For a regular closed set A and a regular open set V containing A, there exists an $\mathcal{I}_{rg}$-open set U of X such that A⊆U⊆cl*(U)⊆V.

(6). For a regular closed set A and a regular open set V containing A, there exists an $\star$-open set U of X such that A⊆U⊆cl*(U)⊆V.

(7). For disjoint regular closed sets A and B, there exist disjoint $\star$-open sets U and V such that A⊆U and B⊆V.

Proof.
(1)⇒(2). Suppose that A and B are disjoint regular closed sets. Since X is mildly normal, there exist disjoint open sets U and V such that A⊆U and B⊆V. But every open set is an $\mathcal{I}_{g\#}$-open set. This proves (2).

(2)⇒(3). The proof follows from the fact that every $\mathcal{I}_{g\#}$-open set is an $\mathcal{I}_g$-open set.

(3)⇒(4). The proof follows from the fact that every $\mathcal{I}_g$-open set is an $\mathcal{I}_{rg}$-open set.

(4)⇒(5). Suppose A is a regular closed and B is a regular open set containing A. Then A and X−B are disjoint regular closed sets. By hypothesis, there exist disjoint $\mathcal{I}_{rg}$-open sets U and V such that A⊆U and X−B⊆V. Since X−B is regular closed and V is $\mathcal{I}_{rg}$-open, by Lemma 2.20, X−B⊆int*(V) and so X−int*(V)⊆B. Again, U∩V=ϕ implies that U∩int*(V)=ϕ and so cl*(U)⊆X−int*(V)⊆B. Hence U is the required $\mathcal{I}_{rg}$-open set such that A⊆U⊆cl*(U)⊆B.

(5)⇒(6). Let A be a regular closed set and V be a regular open set containing A. Then there exists an $\mathcal{I}_{rg}$-open set G of X such that A⊆G⊆cl*(G)⊆V. By Lemma 2.20, A⊆int*(G). If U=int*(G), then U is an $\star$-open set and A⊆U⊆cl*(U)⊆cl*(G)⊆V. Therefore, A⊆U⊆cl*(U)⊆V.

(6)⇒(7). Let A and B be disjoint regular closed subsets of X. Then X−B is a regular open set containing A. By hypothesis, there exists an ⋆-open set U of X such that A⊆U⊆cl⋆(U)⊆X−B. If V=X−cl⋆(U), then U and V are disjoint ⋆-open sets of X such that A⊆U and B⊆V.

(7)⇒(1). Let A and B be disjoint regular closed sets of X. Then there exist disjoint ⋆-open sets U and V such that A⊆U and B⊆V. Since $\mathcal{I}$ is completely codense, by Lemma 1.1, $\tau^{*}\subseteq\tau^{\alpha}$ and so U, V∈τ^{α}. Hence A⊆U⊆int(cl(int(U)))=G and B⊆V⊆int(cl(int(V)))=H. G and H are the required disjoint open sets containing A and B respectively. This proves (1). □

If $\mathcal{I}$=$\mathcal{N}$, in the above Theorem 2.21, then $\mathcal{I}_{rg}$-closed sets coincide with rαg-closed sets and so we have the following Corollary 2.22.

Corollary 2.22. *Let (X, τ) be a space. Then the following are equivalent.*

(1). X is mildly normal.

(2). For disjoint regular closed sets A and B, there exist disjoint $\alpha g^{\#}$-open sets U and V such that A⊆U and B⊆V.

(3). For disjoint regular closed sets A and B, there exist disjoint αg-open sets U and V such that A⊆U and B⊆V.

(4). For disjoint regular closed sets A and B, there exist disjoint rαg-open sets U and V such that A⊆U and B⊆V.

(5). For a regular closed set A and a regular open set V containing A, there exists an rαg-open set U of X such that A⊆U⊆cl_{α}(U)⊆V.

(6). For a regular closed set A and a regular open set V containing A, there exists an α-open set U of X such that A⊆U⊆cl_{α}(U)⊆V.

(7). For disjoint regular closed sets A and B, there exist disjoint α-open sets U and V such that A⊆U and B⊆V.

If $\mathcal{I}$={ϕ} in the above Theorem 2.21, we get the following Corollary 2.23.

Corollary 2.23. *Let (X, τ) be a space. Then the following are equivalent.*

(1). X is mildly normal.

(2). For disjoint regular closed sets A and B, there exist disjoint $g^{\#}$-open sets U and V such that A⊆U and B⊆V.

(3). For disjoint regular closed sets A and B, there exist disjoint g-open sets U and V such that A⊆U and B⊆V.

(4). For disjoint regular closed sets A and B, there exist disjoint rg-open sets U and V such that A⊆U and B⊆V.

(5). For a regular closed set A and a regular open set V containing A, there exists an rg-open set U of X such that A⊆U⊆cl(U)⊆V.

(6). For a regular closed set A and a regular open set V containing A, there exists an open set U of X such that A⊆U⊆cl(U)⊆V.

(7). For disjoint regular closed sets A and B, there exist disjoint open sets U and V such that A⊆U and B⊆V.

3. $\mathcal{I}_{g\#}$-regular Spaces

An ideal topological space (X, τ, $\mathcal{I}$) is said to be an $\mathcal{I}_{g\#}$-regular space if for each pair consisting of a point x and a closed set B not containing x, there exist disjoint $\mathcal{I}_{g\#}$-open sets U and V such that x∈U and B⊆V. Every regular space is $\mathcal{I}_{g\#}$-regular, since every open set is $\mathcal{I}_{g\#}$-open. The following Example 3.1 shows that an $\mathcal{I}_{g\#}$-regular space need not be regular. Theorem 3.2 gives a characterization of $\mathcal{I}_{g\#}$-regular spaces.

Example 3.1. *Consider the ideal topological space (X, τ, $\mathcal{I}$) of Example 2.1. Then ϕ*=ϕ, ({b})*=ϕ, ({a, b})*={a}, ({b, c})*={c} and X*={a, c}. Since every αg-open set is $\star$-closed, every subset of X is $\mathcal{I}_{g\#}$-closed and so every subset of X is $\mathcal{I}_{g\#}$-open. This implies that (X, τ, $\mathcal{I}$) is $\mathcal{I}_{g\#}$-regular. Now, {c} is a closed set not containing a∈X, {c} and a are not separated by disjoint open sets. So (X, τ, $\mathcal{I}$) is not regular.*

Theorem 3.2. *In an ideal topological space (X, τ, $\mathcal{I}$), the following are equivalent.*

(1). X is $\mathcal{I}_{g\#}$-regular.

(2). For every open set V containing x∈X, there exists an $\mathcal{I}_{g\#}$-open set U of X such that x∈U⊆cl*(U)⊆V.

Proof.
(1)⇒(2). Let V be an open subset such that x∈V. Then X−V is a closed set not containing x. Therefore, there exist disjoint $\mathcal{I}_{g\#}$-open sets U and W such that x∈U and X−V⊆W. Now, X−V⊆W implies that X−V⊆int*(W) and so X−int*(W)⊆V. Again, U∩W=ϕ implies that U∩int*(W)=ϕ and so cl*(U)⊆X−int*(W). Therefore, x∈U⊆cl*(U)⊆V. This proves (2).

(2)⇒(1). Let B be a closed set not containing x. By hypothesis, there exists an $\mathcal{I}_{g\#}$-open set U such that x∈U⊆cl*(U)⊆X−B. If W=X−cl*(U), then U and W are disjoint $\mathcal{I}_{g\#}$-open sets such that x∈U and B⊆W. This proves (1). □

Theorem 3.3. *If (X, τ, $\mathcal{I}$) is an $\mathcal{I}_{g\#}$-regular, T_1-space where $\mathcal{I}$ is completely codense, then X is regular.*

Proof. Let B be a closed set not containing x∈X. By Theorem 3.2, there exists an $\mathcal{I}_{g\#}$-open set U of X such that x∈U⊆cl*(U)⊆X−B. Since X is a T_1-space, {x} is αg-closed and so {x}⊆int*(U), by Lemma 1.5. Since $\mathcal{I}$ is completely codense, $\tau^*\subseteq\tau^\alpha$ and so int*(U) and X−cl*(U) are α-open sets. Now, x∈int*(U)⊆int(cl(int(int*(U))))=G and B⊆X−cl*(U)⊆int(cl(int(X−cl*(U))))=H. Then G and H are disjoint open sets containing x and B respectively. Therefore, X is regular. □

If $\mathcal{I}$=$\mathcal{N}$ in Theorem 3.2, then we have the following Corollary 3.4 which gives characterizations of regular spaces, the proof of which follows from Theorem 3.3.

Corollary 3.4. *If (X, τ) is a T_1-space, then the following are equivalent.*

(1). X is regular.

(2). For every open set V containing x∈X, there exists an $\alpha g^\#$-open set U of X such that x∈U⊆cl_α(U)⊆V.

If $\mathcal{I}$={ϕ} in Theorem 3.2, then we have the following Corollary 3.5 which gives characterizations of regular spaces, the proof of which follows from Theorem 3.3.

Corollary 3.5. *If (X, τ) is a T_1-space, then the following are equivalent.*

(1). X is regular.

(2). For every open set V containing x∈X, there exists a $g^\#$-open set U of X such that $x \in U \subseteq cl(U) \subseteq V$.

Theorem 3.6. *If every αg-open subset of an ideal topological space (X, τ, $\mathcal{I}$) is $\star$-closed, then (X, τ, $\mathcal{I}$) is $\mathcal{I}_{g^\#}$-regular.*

Proof. Suppose every αg-open subset of X is $\star$-closed. Then by Lemma 1.6, every subset of X is $\mathcal{I}_{g^\#}$-closed and hence every subset of X is $\mathcal{I}_{g^\#}$-open. If B is a closed set not containing x, then {x} and B are the required disjoint $\mathcal{I}_{g^\#}$-open sets containing x and B respectively. Therefore, (X, τ, $\mathcal{I}$) is $\mathcal{I}_{g^\#}$-regular. □

The following Example 3.7 shows that the reverse direction of the above Theorem 3.6 is not true.

Example 3.7. *Consider the real line $\mathcal{R}$ with the usual topology with $\mathcal{I}=\{\phi\}$. Since $\mathcal{R}$ is regular, $\mathcal{R}$ is $\mathcal{I}_{g^\#}$-regular. Obviously U = (0,1) is αg-open being open in $\mathcal{R}$. But U is not $\star$-closed because, when $\mathcal{I} = \{\phi\}$, $cl^*(U) = cl(U) = [0,1] \neq U$.*

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