International Journal of Current Research in Science and Technology
Volume 2, Issue 10 (2016), 1–7.
ISSN: 2394-5745
Available Online: http://ijcrst.in/

International Journal of Current Research in Science and Technology

# The Lower Bound Estimation for the Number of Zeros of Random Transcendental Polynomial

**Research Article**

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**Abstract:** The object of this paper is to find a lower bound estimation for the number of zeros of the random transcendental equation $\sum_{v=0}^{n} d_v \xi_v(\omega) z^v = 0$, subject to the condition that the coefficients are non-identically distributed dependent random variables. Throughout the paper n is considered to be very large and $\mu$'s denote positive constants assuming different values in different occurrences.

**MSC:** 60BXX.

**Keywords:** Probability spaces, Characteristic function, Joint density function, Moment matrix, Exceptional set, Marginal frequency function, Distribution function.



## 1. Introduction

$N_n(\omega)$ is the number of zeros of the random transcendental Polynomial

$$f_n(z,\ \omega) = \sum_{v=0}^{n} d_v \xi_v(\omega) z^v \tag{1}$$

and $d_v$'s be non-zero real numbers, when $\xi_v(\omega)$'s are symmetric stable variates with characteristic function

$$\exp\left(-C\,|t|^{\alpha}\right),\ C>0,\ 1<\alpha\leq 2.$$

Assuming the coefficients $\xi_v(\omega)$'s are non-identically distributed dependent random variables on probability space $(\Omega,\ B,\ P)$. Define Normal Distribution with mean zero and joint density function

$$M^{\frac{1}{2}}(2\pi)^{\frac{-n}{2}} exp\left(\frac{-1}{2}\overline{a}'M\overline{a}\right), \qquad \overline{a}' = [\xi_1(\omega),\ \xi_2(\omega),\ \ldots,\ \xi_n(\omega)] \tag{2}$$

where $M^{-1}$ is the moment matrix with $\sigma_i = 1$, $\rho_{ij} = \rho$, $0<\rho<1$, $i\neq j$, $i,\ j = 0,\ 1, \ldots,\ n$ and $\overline{a}$ is the column vector whose transpose is $\overline{a}'$ follows from [1], [2], [3]. Let G be the exceptional set defined by $G = \left\{\omega | N_n(\omega) > \mu(\log\log\ n)^2 \log\ n\right\}$, where $\mu$ is a positive constant. We introduce a notation $\lambda = \log\ n$ and M be the integer defined by $M = \left[\alpha\lambda\frac{k_n}{t_n}\right]+1$, where $\alpha$ is a positive constant and [.] implies the greatest integer function. Let k be the integer determined by $M^{2k} \leq n < M^{2k+2}$.

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Again we introduce a new notation $\lambda_m = m^{\frac{1}{2}} \log\ n$ the random algebraic polynomial and $M_m$ be a sequence of integers defined by $M_m = \left[b\left(\frac{k_n}{t_n}\right)\log\ n\right]+1$, $m = 1,\ 2, \ldots$, where b is a positive constant. Let k be the integer determined by

$$(2k)!\ M_n^{2k} \leq n < (2k+2)!\ M_n^{2k+2}.$$

We shall use the fact that each $\xi_v(\omega)$ has marginal frequency function $\frac{1}{2\pi} exp\left(-\frac{\omega^2}{2}\right)$ . In this paper we have established two theorems where the second theorem is a modified and more effective interpretation of the first one in the sense of Evans [1], since the exceptional set G obtained in this case is independent of n. Throughout this paper n is considered to be very large and $\mu$'s, $\alpha$'s, b's denote positive constants assuming different values in different occurrences.

## 2. Preliminary and Some Results

**Theorem 2.1.** *Let* $f_n(z,\ \omega) = \sum_{v=0}^{n} d_v \xi_v(\omega) z^v$ *be a random transcendental polynomial, where the* $\xi_v(\omega)$*'s are non-identically distributed dependent random variables with mean zero and joint density function given by* (2). *Let* $d_v$*'s be non-zero real numbers such that*

$$\frac{k_n}{t_n} = 0(\log\ n),\ \textit{where}\ k_n = \max_{0\leq v\leq n} |d_v|,\ t_n = \min_{0\leq v\leq n} |d_v|$$

*Then there exist a positive integer* $n_0$ *Such that for* $n > n_0$

$$N_n(\omega)\ \geq\ \mu\ \frac{\log\ n}{\log\left(\frac{k_n}{t_n}\right)\log\ n}$$

*and*

$$P(G)\ \leq\ \mu'\ \frac{log\left(\frac{k_n}{t_n}\right)\log\ n}{\log\ n}.$$

*Where* $N_n(\omega)$ *is the number of real zeros of the polynomial and* $G$ *is the exceptional set by* $G = \left\{\omega | N_n(\omega) > \mu(\log\log\ n)^2 \log\ n\right\}$ *and* $P(G) \to 0$ *as* $n \to \infty$.

*Proof.* Let

$$\lambda = \log\ n \tag{3}$$

and M be the integer defined by

$$M = \left[\alpha\lambda\frac{k_n}{t_n}\right]+1 \tag{4}$$

where $\alpha$ is a positive constant and [.] implies the greatest integer function. Let k be the integer determined by

$$M^{2k} \leq n < M^{2k+2} \tag{5}$$

It follows from (3), (4), and (5) that, for two constants $\mu_1$ and $\mu_2$,

$$\mu_1\ \frac{\log\ n}{\log\left(\frac{k_n}{t_n}\right)\log\ n} \leq k \leq \mu_2 \frac{\log\ n}{\log\left(\frac{k_n}{t_n}\right)\log\ n} \tag{6}$$

We shall consider $f_n(z,\ \omega)$ at the points

$$z_m = (1 - M^{-2m})^{\frac{1}{2}} \tag{7}$$

for $m=\left[\frac{k}{2}\right]+1,\ \left[\frac{k}{2}\right]+2,\ldots$,k. Let

$$f_n\left(z_m,\,\omega\right)=\sum_1 d_v\xi_v\left(\omega\right)z_m^v+\left(\sum_2+\sum_3\right)d_v\xi_v\left(\omega\right)z_m^v v=\ A_m\left(\omega\right)+R_m\left(\omega\right)$$

where v ranges from $M^{2m-1}+1$ to $M^{2m+1} in \sum_1$ and from 0 to $M^{2m-1}$ in $\sum_2$ and from $M^{2m+1}+1$ to n in $\sum_3$ where

$$A_m\left(\omega\right)=\sum_1 d_v\xi_v\left(\omega\right)z_m^v \text{ and}$$

$$R_m\left(\omega\right)=\left(\sum_2+\sum_3\right)d_v\xi_v\left(\omega\right)z_m^v$$

The following lemmas are necessary for the rest proof of Theorem 2.1

**Lemma 2.2.** *For* $\alpha_1>0,\ \sigma_m\ >\ \alpha_1 t_n M^{2m}$, *where*

$$\sigma_m{}^2=(1-\rho)\sum_1 d_v^2 z_m^{2v}+\rho\left(\sum_1 d_v z_m^v\right)^2,\ (0<\rho<1) \tag{*}$$

*Proof.*

$$\sum_1 d_v z_m^v>t_n\sum_1 z_m^v>t_n M^{2m}\left(\frac{B}{A\sqrt{e}}\right)\Rightarrow\left(\sum_1 d_v z_m^v\right)^2>\alpha_1^2 t_n^2 M^{4m}, \tag{i}$$

$\alpha_1$ is a positive constant. Where A, B and constants satisfying the relations, $A>1$ and $0<B<1$. Again

$$\sum_1 d_v^2 z_m^{2v}>M^{2m}t_n^2\left(\frac{B}{Ae}\right) \tag{ii}$$

From (i) and (ii), (*) becomes $\sigma_m^2>\alpha_1^2 t_n^2 M^{4m}$ (where$\alpha_1$ is a positive constant) and hence $\sigma_m>\alpha_1 t_n M^{2m}$. Which gives the result.

**Lemma 2.3.**

$$P\left\{\omega:\left|\sum_2 d_v\xi_v\left(\omega\right)z_m^v\right|>\lambda\widetilde{\sigma}_m\right\}<\sqrt{\frac{2}{\pi}}\frac{e^{\frac{-\lambda^2}{2}}}{\lambda}\quad where\ \ \widetilde{\sigma}_m^2=(1-\rho)\sum_2 d_v^2 z_m^{2v}+\rho\left(\sum_2 d_v z_m^v\right)^2,\ (0<\rho<1).$$

*Proof.* Let $F(z)$ be the distribution function of $\sum_2 d_v\xi_v\left(\omega\right)z_m^v$ Then

$$P\left\{\omega:\left|\sum_2 d_v\xi_v\left(\omega\right)z_m^v\right|>\lambda\widetilde{\sigma}_m\right\}=1-\left\{F\left(\lambda\widetilde{\sigma}_m\right)-F\left(-\lambda\widetilde{\sigma}_m\right)\right\}$$

$$=\sqrt{\frac{2}{\pi}}\int_\lambda^\infty e^{\frac{-t^2}{2}}dt<\sqrt{\frac{2}{\pi}}\frac{e^{\frac{-\lambda^2}{2}}}{\lambda}$$

**Lemma 2.4.**

$$P\left\{\omega:\left|\sum_3 d_v\xi_v\left(\omega\right)z_m^v\right|>\ \lambda\ \widetilde{\widetilde{\sigma}}_m\right\}<\sqrt{\frac{2}{\pi}}\frac{e^{\frac{-\lambda^2}{2}}}{\lambda},\ \ where\ \widetilde{\widetilde{\sigma}}_m^2=(1-\rho)\sum_3 d_v^2 z_m^{2v}+\rho\left(\sum_3 d_v z_m^v\right)^2,\ (0<\rho<1)$$

**Lemma 2.5.** *For a fixed m,*

$$P\left\{\omega\text{:}\left|R_m\left(\omega\right)\right|<\sigma_m\right\}>1-2\sqrt{\frac{2}{\pi}}\frac{e^{\frac{-\lambda^2}{2}}}{\lambda}$$

*Proof.* For given m, we have $|R_m(\omega)| < \lambda\left(\widetilde{\sigma}_m + \widetilde{\widetilde{\sigma}}_m\right)$ again

$$\sum_2 d_v^2 z_m^{2v} \leq 2k_n^2 M^{2m-1} \; and \; \sum_2 d_v z_m^v \leq 2k_n M^{2m-1}$$

Hence $\widetilde{\sigma}_m^2 \leq \alpha_2^2 k_n^2 M^{2m-1} < \alpha_2^2 k_n^2 M^{4m-2}, \alpha_2 > 0$. Similarly $\widetilde{\widetilde{\sigma}}_m^2 \leq \alpha_3^2 k_n^2 M^{2m-1} < \alpha_3^2 k_n^2 M^{4m-2}$, $\alpha_3 > 0$. Thus

$$\begin{aligned} |R_m(\omega)| &< \lambda(\alpha_2+\alpha_3) k_n M^{2m-1} \\ &< \frac{\left\{\lambda\left(\frac{\alpha_2+\alpha_3}{\alpha_1}\right)\frac{k_n}{t_n}\sigma_m\right\}}{M} \quad \text{by Lemma 2.1} \\ &< \sigma_m \text{ by the definition of M} \end{aligned}$$

Since the distribution function of $A_m(\omega)$ is

$$\frac{1}{\sqrt{2\pi}\,\sigma_m}\int_{-\infty}^{z} exp\left[-\left(\frac{t^2}{2\sigma_m^2}\right)\right] dt$$

where

$$\sigma_m^2 = (1-\rho)\sum_1 d_v^2 z_m^{2v} + \rho\left(\sum_1 d_v z_m^v\right)^2, \; 0<\rho<1$$

The distribution function of $\frac{A_m}{\sigma_m}$ is

$$\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{z} \exp\left[-\left(\frac{t^2}{2}\right)\right] dt = D(z) \;\; (say)$$

Now let us define random events $E_m$ and $F_m$ by

$$E_m = \{\omega : A_{2m}(\omega) \geq \sigma_{2m}, \; A_{2m+1}(\omega) < -\sigma_{2m-1}\}$$

$$F_m = \{\omega : A_{2m}(\omega) < -\sigma_{2m}, \; A_{2m+1}(\omega) \geq \sigma_{2m+1}\}$$

It can be easily seen that $P(E_m \cup F_m) > \delta > 0$, where $\delta$ is a positive constant. Then proceeding exactly as Samal and Mishra [3] we shall get the following results.

$$N_n(\omega) > \mu_1 k \geq \frac{\mu \log\ n}{\log\left(\frac{k_n}{t_n}\right)\log\ n} \quad \text{by(6)}$$

and

$$P(G) < \mu_1 k \frac{1}{\lambda e^{\frac{\lambda^2}{2}}} + \frac{\mu_2}{k} \leq \mu' \left\{\frac{\log\left(\frac{k_n}{t_n}\right)\log\ n}{\log n}\right\}$$

Since $\frac{k_n}{t_n} = 0(\log\ n)$, $P(G) \to 0 \; as \; n \to \infty$. Hence the Theorem 2.1. □

**Theorem 2.6.** *Let*

$$f_n(z,\omega) = \sum_{v=0}^{n} d_v \xi_v(\omega) z^v$$

*be a random transcendental polynomial, where the $\xi_v(\omega)$'s are non-identically distributed dependent random variables with mean zero and joint density function given by (2). Let $d_v$'s be non-zero real numbers, such that*

$$\frac{k_n}{t_n} = 0(\log\ n), \; where \; k_n = \max_{0\leq v\leq n}|d_v|, \; t_n = \min_{0\leq v\leq n}|d_v|$$

*Then there exist a positive integer $n_0$ such that for $n > n_0$*

$$N_n(\omega) \geq \mu \frac{\log\ n}{\log\left(\frac{k_n}{t_n}\log\ n\right)} \quad and$$

$$P(G) \leq \mu'\left\{\frac{log\left(log\left(\frac{k_{n_0}}{t_{n_0}}\right)logn_0\right)}{logn_0}\right\}^{\frac{1}{2}}$$

*$N_n(\omega)$ is the number of real zeros of the polynomial and $G$ is the exceptional set.*

*Proof.* Let

$$\lambda_m = m^{\frac{1}{2}}\log\ n \tag{8}$$

and $M_m$ be a sequence of integers defined by

$$M_m = \left[b\left(\frac{k_n}{t_n}\right)\log\ n\right] + 1, \ \ m = 1,\ 2, \ldots \tag{9}$$

where b is a positive constant. Let k be the integer determined by

$$(2k)!\ M_n^{2k} \leq n < (2k+2)!\ M_n^{2k+2} \tag{10}$$

It follows from (9) and (10) that for two constants $\mu_1$ and $\mu_2$

$$\mu_1 \frac{\log\ n}{\log\left(\frac{k_n}{t_n}\log\ n\right)} \leq k \leq \mu_2 \frac{\log\ n}{\log\left(\frac{k_n}{t_n}\log\ n\right)} \tag{11}$$

We consider $f_n(z, \omega)$ at the points

$$z_m = \left\{1 - \frac{1}{(2k)!\ M_m^{2m}}\right\}^{\frac{1}{2}} \tag{12}$$

$for\ m = \left[\frac{k}{2}\right]+1,\ \left[\frac{k}{2}\right]+2, \ldots\ldots., k.$ for large n we write

$$f_n(z_m, \omega) = A_m(\omega) + R_m(\omega)$$

where

$$A_m(\omega) = \sum_1 d_v \xi_v(\omega) z_m^v$$

$$and \qquad R_m(\omega) = \left(\sum_2 + \sum_3\right) d_v \xi_v(\omega) z_m^v$$

and the index v ranges from $(2m-1)!\ M_m^{2m-1}+1$ to $(2m+1)!\ M_m^{2m+1}$ in $\sum_1$ and from 0 to $(2m-1)!\ M_m^{2m-1}$ in $\sum_2$ and from $(2m+1)!\ M_m^{2m+1}+1$ to n in $\sum_3$

The following lemmas are necessary for the rest proof of Theorem 2.6.

**Lemma 2.7.** *For $r_1 > 0$, $\sigma_m > r_1 t_n (2m)!\ M_m^{2m}$, where*

$$\sigma_m^2 = (1-\rho)\sum_1 d_v^2 z_m^{2v} + \rho\left(\sum_1 d_v z_m^v\right)^2, \ 0 < \rho < 1 \tag{*}$$

*Proof.*

$$\sum_1 d_v^2 z_m^{2v} > t_n^2 \left( (2m)!M_m^{2m} \frac{B}{Ae} \right)^2, \quad where\ A > 1,\ 0 < B < 1 \tag{i}$$

and

$$\left( \sum_1 d_v z_m^v \right)^2 > t_n^2 \left( \sum_1 z_m^v \right)^2 > t_n^2 \left( (2m)!M_m^{2m} \frac{B}{A\sqrt{e}} \right)^2 \tag{ii}$$

From (i) and (ii), (*) becomes

$$\sigma_m^2 > (1-\rho)\, t_n^2 \left( (2m)!M_m^{2m} \frac{B}{Ae} \right)^2 + \rho t_n^2 \left( (2m)!M_m^{2m} \frac{B}{A\sqrt{e}} \right)^2$$

And hence

$$\sigma_m > r_1 t_n (2m)!\ M_m^{2m}$$

Again the following two Lemmas 2.8 and 2.9 can be proved as the process that has been adopted in Theorem 2.1.

**Lemma 2.8.**

$$P\left\{ \omega\colon \left| \sum_2 d_v \xi_v(\omega)\, z_m^v \right| < \lambda q_m \right\} < \sqrt{\frac{2}{\pi}} \frac{e^{\frac{-\lambda^2}{2}}}{\lambda} \quad where\ \ {q_m}^2 = (1-\rho) \sum_2 d_v^2 z_m^{2v} + \rho \left( \sum_2 d_v z_m^v \right)^2.$$

**Lemma 2.9.**

$$P\left\{ \omega\colon \left| \sum_3 d_v \xi_v(\omega)\, z_m^v \right| < \lambda Q_m^2 \right\} < \sqrt{\frac{2}{\pi}} \frac{e^{\frac{-\lambda^2}{2}}}{\lambda} \quad where\ \ Q_m^2 = (1-\rho) \sum_3 d_v^2 z_m^{2v} + \rho \left( \sum_3 d_v z_m^v \right)^2.$$

**Lemma 2.10.** *For a fixed m,* $P\{\ \omega\colon\ |R_m(\omega)| < \sigma_m\} > 1 - 2\sqrt{\frac{2}{\pi}} \frac{e^{\frac{-\lambda^2}{2}}}{\lambda}$.

*Proof.* For a given m, we have $|R_m(\omega)| < \lambda(q_m + Q_m)$. Now

$$\sum_2 d_v^2 z_m^{2v} \le 2k_n^2 (2m-1)!\ M_m^{2m-1} \ \text{ and}$$

$$\sum_2 d_v z_m^v \le 2k_n\ (2m-1)!\ M_m^{2m-1}$$

${q_m}^2 \le r_2 k_n (2m-1)!\ M_m^{2m-1}$, where $r_2$ is a positive constant $< r_2 k_n (2m)!\ M_m^{2m-1}$. Similarly $Q_m^2 < r_3 k_n (2m)!\ M_m^{2m-1}$, where $r_3$ is a positive constant. So,

$$|R_m(\omega)| < \lambda k_n \frac{M_m^{2m}(r_2 + r_3)}{M_m} < \sigma_m.$$

Therefore $|R_m(\omega)| < \sigma_m$ except for a set of measure at most

$$2\sqrt{\frac{2}{\pi}} \frac{e^{\frac{-\lambda^2}{2}}}{\lambda} \qquad (by\ definition\ of\ M_m)$$

$$for\ m = \left[ \frac{k}{2} \right] + 1,\ \left[ \frac{k}{2} \right] + 2, \ldots, k.$$

Now defining the $E_m$ and $F_m$ as in Theorem 2.1, we can have $P(E_m U F_m) > \delta > 0$, where $\delta$ is an absolute constant. Now let us define random variables $X_m(\omega)$, $Y_m(\omega)$ and net $\mathrm{G}_m$ as

$$X_m(\omega) = \begin{cases} 1, & \text{if } \omega \in E_m \cup F_m \\ 0, & \text{if } \omega \in (E_m \cup F_m)'. \end{cases}$$

Thus

$$P\{\ \omega{:}X_m\left(\omega\right)=1\}=\delta$$

and

$$P\{\ \omega{:}X_m\left(\omega\right)=0\}=1-\delta$$

Let $G_m=\{\ \omega{:}\left|R_{2m}\left(\omega\right)\right|<\sigma_{2m}\ and\ R_{2m+1}\left(\omega\right)<\sigma_{2m+1}\}$ and

$$Y_m\left(\omega\right)=\begin{cases}0, & \text{if } \omega\in G_m;\\ 1, & \text{if } \omega\in\left(G_m\right)'.\end{cases}$$

Let $T_m\left(\omega\right)=X_m\left(\omega\right)-X_m\left(\omega\right)Y_m\left(\omega\right)$. If $T_m\left(\omega\right)=1$, *then there exist a zero of the polynomial in the interval* $\left(z_{2m},\ z_{2m+1}\right)$. Now proceeding as Samal and Mishra [3] we get,

$$\begin{aligned} N_n(\omega) &\geq \mu_1 k \\ &\geq \mu\frac{\log\ n}{\log\left(\frac{k_n}{t_n}\log n\right)} && \textit{by } (11) \end{aligned}$$

$$\begin{aligned} \text{and}\qquad P\left(G\right) &\leq \frac{\mu_2}{k_0}+\mu_3\sum_{k\geq 2k_0-1}\frac{exp\left(-\frac{\lambda^2_{m_0}}{2}\right)}{\lambda_{m_0}} \\ &\leq \frac{\mu_2}{k_0}+\mu_4\sum_{k\geq 2k_0-1}\frac{1}{\lambda^3_{m_0}} \\ &\leq \frac{\mu_2}{k_0}+2\mu_4\sum_{k\geq k_0}\frac{1}{k^{\frac{3}{2}}} \\ &\leq \frac{\mu_2}{k_0}+2\mu_4\left(\frac{2}{k_0^{\frac{1}{2}}}\right) \\ &\leq \mu'\left\{\frac{\log\left(\log\left(\frac{k_{n_0}}{t_{n_0}}\right)\log n_0\right)}{\log n_0}\right\}^{\frac{1}{2}} && \textit{by } (11) \end{aligned}$$

hence the Theorem 2.6. □

## References


[1] E.A.Evans, *On the number of a random algebraic equation*, Proc. London Math. Soc., 3(15)(1965), 731-749.

[2] G.Samal, *Onthe number of real roots of a random algebraic equation*, Proc. Cambridge Philos. Soc., 58(1962), 433-442.

[3] G.Samal and M.N.Mishra, *Onthe lower bound of the number of real roots of a randomalgebraic with infinite variance*, Proc. Am. Math. Soc., 33(1972), 523-528.

[4] M.Sambandham, *On the upper bound of the number of real zeros of a random algebraicequation*, J. Indian Math. Soc., 42(1978), 15-26.