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Results on Scrambled Sets of Full Measurable Functions

Research Article

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Abstract: In this paper we constructed a function f possessing a scrambled set of full measure, then the function g is strictly increasing on [0, 1/2] and strictly decreasing on [1/2, 1] and further we extend our result that $\lim \sup_{n\to\infty} |f^n(x) - f^n(y)| = 1$ and $\lim \inf_{n\to\infty} |f^n(x) - f^n(y)| = 0$.These claims to extend results such a kind obtained by Brukner A.M and [3]. Simtal .J [10] and Jiehu Mai [8].

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1. Introduction

Let $f : [0, 1] \to [0, 1]$ is defined by $f(x) = \begin{cases} 2x & if\ 0 \le x \le 1/2 \\ 2-2x & if\ \ 1/2 \le x \le 1 \end{cases}$. And also we define that

(a). If $x = 0.0x_2.x_3.x_4\ldots$ (binary expansion), then $f(x) = 0.x_2.x_3.x_4\ldots$

(b). If $x = 0.1x_2.x_3.x_4\ldots$ (binary expansion), then $f(x) = 0.y_2.y_3.y_4\ldots$, where $y_i = 0$ if $x_i = 0$ and $y_i = 1$ if $x_i = 0$.

(c). If every finite sequence of 0's and 1's appears in the binary expansion of x, the x has a dense orbit. Thus almost every $x \in [0, 1]$ has a dense orbit.

Smital .J [10] has shown that f has a scrambled set of full outer measure, but for any measurable scrambled set for f must have zero measure. We first show that the function f has scrambled set S of Borel type$G_{\delta\sigma}$ that has a cardinality of the continuum in every interval. A suitable transformation then the results in the function g. Our main aim is obtaining S arises from the following considerations. If S is a scrambled set of f and $x \neq y$ are in S, then the binary expansions of x and y must contain arbitrary long string of 0's and 1's from definitions of (a) and (b) above. These strings must be positioned properly so that the definition of scrambled set satisfied for x and y. These requirements must be achieved by every pair (x,y) in S. We shall achieve this by representing certain points in I by wedges.

2. Main Result

Theorem 2.1. *Let* $S = \{x_\omega : \omega \in W\}$

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(1). S is c-dense.

(2). S is a scrambled set for the tent map.

(3). S is first category $G_{\delta\sigma}$ subset of [0, 1].

Proof.

(1). (i). card S=c

(ii). For any finite of integer with $i_1 < i_2 < \dots < i_n$ there exists $\omega \in W$ containing $l_{i_1}, l_{i_2}, \dots, l_{i_n}$. Corresponding to $i_1, i_2, \dots, i_n$, but no other lattice point l_j with $j \le i_n$. It follows that S is dense.

(2). The tent map $f : [0, 1] \to [1, 0]$ is defined by $f(x) = \begin{cases} 2x, & \text{if } 0 \le x \le 1/2; \\ 2 - 2x, & \text{if } 1/2 \le x \le 1. \end{cases}$

If $\omega_1, \omega_2 \in W$ with $\omega_1 \neq \omega_2$ and $x = x_{\omega_1}, y = x_{\omega_2}$ then there exists arbitrarily long finite sequences of consecutive integers $n, n+1, \dots, N$ and $m, m+1, \dots, M$ such that $x_i = y_i = 0$ for $i = n, n+1, \dots, N$ and $x_i = 1$ and $y_i = 0$ (or $x_i = 0$ and $y_i = 1$) for $i = m, m+1, \dots, M$ and such that x_{m-1}, y_{m-1} are either both 0 or both 1. Using the property of result (1), We have

$$\begin{aligned}
|f^n(x) - f^n(y)| &= |0.x_{n+1}.x_{n+2} \dots x_N x_{N+1} \dots - 0.y_{n+1}.y_{n+2} \dots y_N.y_{N+1} \dots| \\
&= \left| \frac{x_{n+1}}{2^1} + \frac{x_{n+2}}{2^2} + \dots \frac{x_N}{2^{N-n}} + \frac{x_{N+1}}{2^{N-n+1}} + \dots - \left(\frac{y_{n+1}}{2} + \frac{y_{n+2}}{2^2} + \dots \right) \right| \\
&\le \frac{1}{2} |x_{n+1} - y_{n+1}| + \frac{1}{2^2} |x_{n+2} - y_{n+2}| + \dots + \frac{1}{2^{N-n}} |x_N - y_N| + \frac{1}{2^{N-n+1}} |x_{N+1} - y_{N+1}| + \dots \\
&\le \frac{1}{2^{N-n+1}} + \frac{1}{2^{N-n+2}} + \frac{1}{2^{N-n+3}} + \dots \\
&= \frac{1}{2^{N-n+1}} \left[1 + \frac{1}{2^1} + \frac{1}{2^2} + \dots \right] \\
&= \frac{1}{2^{N-n}}.
\end{aligned}$$

Therefore , $\lim_{n\to\infty} \inf |f^n(x) - f^n(y)| = 0$.

$$\begin{aligned}
&= |0.x_{m+1}.x_{m+2} \dots x_M.x_{M+1} \dots - 0.y_{m+1}.y_{m+2} \dots y_M.y_{M+1} \dots| \\
&= \left| \frac{x_{m+1}}{2^1} + \frac{x_{m+2}}{2^2} + \dots \frac{x_M}{2^{M-m+1}} + \dots - \left(\frac{y_{m+1}}{2^1} + \frac{y_{m+2}}{2^2} + \dots \right) \right| \\
&\ge \frac{1}{2^1(x_{m+1} - y_{m+1})} + \frac{1}{2^2(x_{m+2} - y_{m+2})} + \dots \frac{1}{2^{M-m}} (x_M - y_M) \\
&= \frac{1}{2^1} + \frac{1}{2^2} + \dots + \frac{1}{2^{M-m}} \\
&= 1 - \frac{1}{2^{M-m}}.
\end{aligned}$$

Therefore $\lim_{m\to\infty} \sup |f^m(x) - f^m(y)| \ge 1$. But $\lim_{m\to\infty} \sup |f^m(x) - f^m(y)| > 1$ is possible. Hence $\lim_{m\to\infty} \sup |f^m(x) - f^m(y)| = 1$. Using the property of result (2). We can similarly prove

$$\lim_{n\to\infty} \sup |f^n(x) - f^n(y)| = 1 \quad \textit{and} \qquad \lim_{n\to\infty} \inf |f^n(x) - f^n(y)| = 0.$$

Thus S is a scrambled set for f.

(3). Let S_1 consist of those $x \in S$ requiring no more then one wedge in the representations. We define from S_1 to $\Lambda \times \Lambda$. i.e if $x \in S_1$ there exists a unique $r, s \in \Lambda$ such that $x = x_{w_{rs}}$. The map associate to x this $(r, s) \in \Lambda$. Let $A_1 = \Lambda \times \Lambda \, \{(r, s) : r < s\} = \{(r, s) \in \Lambda \times \Lambda : r < s\}$. Then A_1 is a G_δ set of cardinality c in the complete space $\Re^2$. Then $(x_n, s_n) \to (r, s)$ is A_1 if and only if $x_n \to x \in S_1$ where $x_n = x_{w_{r_n s_n}}$ and $x = x_{w_{rs}}$. Therefore S_1 is homeomorphic to A_1. Similarly if

$$S_n = \{x : x = x_w \text{ where } \omega = w_{r_1 s_1} \cup \cdots \cup \omega_{r_n s_n} \text{ with } r_1 < s_1 < \cdots < r_n < s_n \text{ consits of n wedges}\}.$$

Therefore S_n is hiomeomorphic to A_n where $A_n = \{(r_1 s_1 . r_2 s_2 \ldots r_n s_n) : r_1 < s_1 < \cdots < r_n < s_n\}$ and $r_k . s_k \in \Lambda$ for $k = 1, 2, \ldots, n$. Therefore S_n is G_δ set, as $S = \bigcup_{n=1}^{\infty} S_n$. S is $G_{\delta\sigma}$. S_n is now dense because x with long strings of 0,s and 1's alternating in binary expansion cannot be in S_n. Since S is countable union of nowhere dense sets. S is of category. Therefore S is first category $G_{\delta\sigma}$ subset of [0, 1].

□

Theorem 2.2. *There exists a continuous function $g : I \to I$ such that the function g is strictly increasing on [0, 1/2] and strictly decreasing on [1/2, 1] and corresponding scrambled set T has the following properties.*

(1). T is F_σ-set.

(2). $\mu(T) = 1$. where $\mu(T)$ is Lebesgue measure of T.

Proof. We will construct a continuous function g satisfying

(1). g is strictly increasing on [0, 1/2]

(2). g is strictly decreasing on [1/2, 1]

Such that the constructed function g possesses a scrambled set T of full measure .i e, $\mu(T) = 1$. Let L be the lattice of pairs of positive integers. Let $\xi \in (0, 1)$ be a fixed irrational number. Let A be the family of positive, finite slopes of all rays emanating from $(0, \xi)$. Lying in the first quadrant, and containing no points of L. If $r, s \in \Lambda$ with $r < s$, let $\omega_{rs} = \{(x, y) \in L : rx + \xi < y < sx + \xi\}$. Let L be enumerated diagonally $l_1, l_2, \ldots$ as follows (1,1), (1,2), (2,1), (1,3), (2,2), (3,1), (1,4), (2,3),...
Let W be the system of finite union of wedges, Thus an element of W consists of the union of finite number of wedges.

$$\omega_{r_1 s_1}, \omega_{r_2 s_2} \ldots \omega_{r_n s_n}, \qquad r_1 < s_1 < r_2 < s_2 < \ldots r_n < s_n.$$

Now to each $\omega \in W$, ω associate that number $x_\omega \in [0, 1]$ whose binary has a 1 in the i^{th} position if and only if the i^{th} lattice point l_i is in ω. □

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