International Journal of Current Research in Science and Technology

Volume 2, Issue 3 (2016), 1–10.

ISSN: 2394-5745

Available Online: http://ijcrst.in/

International Journal of Current Research in Science and Technology

Symmetry Classifications and Reductions of (2+1)-Dimensional Korteweg-de Vries Equation

Research Article

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Abstract: We establish the symmetry reductions of (2+1)- Dimensional Korteweg-de Vries Equation, $(u_t + u + u^3 u_x + \alpha u_{xxx})_x + \beta u_{yy} = 0$ is subjected to the Lie's classical method. Classification of its symmetry algebra into one- and two-dimensional subalgebras are carried out in order to facilitate its reduction systematically to (1+1)-dimensional PDEs and then to first or second-order ODEs.

MSC: 54A02, 54A40, 54A99, 03F55.

Keywords: Korteweg-de Vries Equation, Symmetry algebra.

1. Introduction

A simple model equation is the Korteweg-de Vries (KdV) equation [4]

$$v_t + 6vv_x + \delta v_{xxx} = 0 \ , \tag{1}$$

which describe the long waves in shallow water. Its modified version is,

$$u_t - 6u^2 u_x + u_{xxx} = 0 \tag{2}$$

and again there is Miura transformation [5]

$$v = u^2 + u_x \ , \tag{3}$$

between the KdV equation (1) and its modified version (2). In 2002, Liu and Yang [4] studied the bifurcation properties of generalized KdV equation (GKdVE)

$$u_t + au^n u_x + u_{xxx} = 0 \ , \quad a \in \mathbb{R} \ , \ n \in \mathbb{Z}^+ \ . \tag{4}$$

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Gungor and Winternitz [8] transformed the Generalized Kadomtsev-Petviashvili Equation (GKPE)

$$(u_t + p(t)uu_x + q(t)u_{xxx})_x + \sigma(y,t)u_{yy} + a(y,t)u_y + b(y,t)u_{xy} + c(y,t)u_{xx} + e(y,t)u_x + f(y,t)u + h(y,t) = 0 \ , \tag{5}$$

to its canonical form and established conditions on the coefficient functions under which (5) has an infinite dimensional symmetry group having a Kac-Moody-Virasoro structure. In [8], they carried out the symmetry analysis of Variable Coefficient Kadomtsev Petviashvili Equation (VCKP) in the form,

$$(u_t + f(x,y,t)uu_x + g(x,y,t)u_{xxx})_x + h(x,y,t)u_y = 0 \ .$$

In this paper, we discuss the symmetry analysis of the (2+1)-dimensional KdV equation

$$(u_t + u + u^3 u_x + \alpha u_{xxx})_x + \beta u_{yy} = 0 \ , \quad \text{where } \alpha, \ \beta \ \in \mathbb{R} \ . \tag{6}$$

Our intention is to show that equation (6) admits a four-dimensional symmetry group and determine the corresponding Lie algebra, classify the one- and two-dimensional subalgebras of the symmetry algebra of (6) in order to reduce (6) to (1+1)-dimensional PDEs and then to ODEs. It is shown that (6) reduces to a once differentiated KdV equation and to a linear PDE $w_{ss}(r,s) = 0$. We shall establish that the symmetry generators form a closed Lie algebra and this allowed us to use the recent method due to Ahmad, Bokhari, Kara and Zaman [**?**] to successively reduce (6) to (1+1)-dimensional PDEs and ODEs with the help of two-dimensional Abelian and non-Abelian solvable subalgebras. This paper is organised as follows: In section 2, we determine the symmetry group of (6) and write down the associated Lie algebra. In section 3, we consider all one-dimensional subalgebras and obtain the corresponding reductions to (1+1)-dimensional PDEs. In section 4, we show that the generators form a closed Lie algebra and use this fact to reduce (6) successively to (1+1)- dimensional PDEs and ODEs. In section 5, we summarises the conclusions of the present work.

2. The Symmetry Group and Lie Algebra of $(u_t + u + u^3 u_x + \alpha u_{xxx})_x + \beta u_{yy} = 0$.

If (6) is invariant under a one parameter Lie group of point transformations (Bluman and Kumei [3], Olver [2])

$$x^* = x + \epsilon \ \xi(x,y,t;u) + O(\epsilon^2) \ , \tag{7}$$

$$y^* = y + \epsilon \ \eta(x,y,t;u) + O(\epsilon^2) \ , \tag{8}$$

$$t^* = t + \epsilon \ \tau(x,y,t;u) + O(\epsilon^2) \ , \tag{9}$$

$$u^* = u + \epsilon \ \phi(x,y,t;u) + O(\epsilon^2) \ , \tag{10}$$

with infinitesimal generator

$$X = \xi(x,y,t;u)\frac{\partial}{\partial x} + \eta(x,y,t;u)\frac{\partial}{\partial y} + \tau(x,y,t;u)\frac{\partial}{\partial t} + \phi(x,y,t;u)\frac{\partial}{\partial u} \tag{11}$$

then the invariant condition is

$$\phi^x + 6u\phi u_x^2 + 6u^2 u_x \phi^x + \phi^{xt} + 3u^2 \phi u_{xx} + u^3 \phi^{xx} + \beta \phi^{yy} + \alpha \phi^{xxxx} = 0. \tag{12}$$

In order to determine the four infinitesimals $\xi,\ \eta,\ \tau$ and ϕ, we prolong V to fourth order. This prolongation is given by the formula

$$\begin{aligned} V^{(4)} &= V + \phi^{x}\frac{\partial}{\partial u_x} + \phi^{y}\frac{\partial}{\partial u_y} + \phi^{t}\frac{\partial}{\partial u_t} + \phi^{xx}\frac{\partial}{\partial u_{xx}} + \phi^{xy}\frac{\partial}{\partial u_{xy}} + \phi^{xt}\frac{\partial}{\partial u_{xt}} \\ &+\phi^{yy}\frac{\partial}{\partial u_{yy}} + \phi^{yt}\frac{\partial}{\partial u_{yt}} + \phi^{tt}\frac{\partial}{\partial u_{tt}} + \phi^{xxx}\frac{\partial}{\partial u_{xxx}} + \phi^{xyy}\frac{\partial}{\partial u_{xyy}} \\ &+\phi^{xxy}\frac{\partial}{\partial u_{xxy}} + \phi^{xtt}\frac{\partial}{\partial u_{xtt}} + \phi^{xyt}\frac{\partial}{\partial u_{xyt}} + \phi^{yyy}\frac{\partial}{\partial u_{yyy}} + \phi^{ttt}\frac{\partial}{\partial u_{ttt}} \\ &+\phi^{xxt}\frac{\partial}{\partial u_{xxt}} + \phi^{yyt}\frac{\partial}{\partial u_{yyt}} + \phi^{ytt}\frac{\partial}{\partial u_{ytt}} + \phi^{xxxx}\frac{\partial}{\partial u_{xxxx}} + \cdots + \phi^{tttt}\frac{\partial}{\partial u_{tttt}} \ . \end{aligned} \tag{13}$$

In the above expression every coefficient of the prolonged generator is a function of x, y, t and u can be determined by the formulae,

$$\phi^{i} = D_i(\phi - \xi u_x - \eta u_y - \tau u_t) + \xi u_{x,i} + \eta u_{y,i} + \tau u_{t,i} \ , \tag{14}$$

$$\phi^{ij} = D_i D_j(\phi - \xi u_x - \eta u_y - \tau u_t) + \xi u_{x,ij} + \eta u_{y,ij} + \tau u_{t,ij} \ , \tag{15}$$

$$\phi^{ijkl} = D_i D_j D_k D_l(\phi - \xi u_x - \eta u_y - \tau u_t) + \xi u_{x,ijkl} + \eta u_{y,ijkl} + \tau u_{t,ijkl} \ , \tag{16}$$

where D_i represents total derivative and subscripts of u derivative with respect to the respective coordinates. To proceed with reductions of Equation (6) we now use symmetry criterion for PDEs. For given equation this criterion is expressed by the formula $V^{(4)}[u_x + 3u^2u_x^2 + u_{xt} + u^3u_{xx} + \beta u_{yy} + \alpha u_{xxxx}] = 0$, whenever, $u_x + 3u^2u_x^2 + u_{xt} + u^3u_{xx} + \beta u_{yy} + \alpha u_{xxxx} = 0$. In (12), we introduced the following quantities:

$$\begin{aligned} \phi^{x} &= D_x(\phi - \xi u_x - \eta u_y - \tau u_t) + \xi u_{xx} + \eta u_{yx} + \tau u_{tx} \\ &= \phi_x + (\phi_u - \xi_x)u_x - \eta_x u_y - \tau_x u_t - \xi_u u_x^2 - \eta_u u_x u_y - \tau_u u_x u_t \ , \\ \phi^{xx} &= D_x D_x(\phi - \xi u_x - \eta u_y - \tau u_t) + \xi u_{xxx} + \eta u_{yxx} + \tau u_{txx} \\ &= \phi_{xx} + (2\phi_{xu} - \xi_{xx})u_x - \eta_{xx}u_y - \tau_{xx}u_t + (\phi_u - 2\xi_x)u_{xx} - 2\eta_x u_{xy} \\ &\quad -2\tau_x u_{xt} + (\phi_{uu} - 2\xi_{ux})u_x^2 - 2\eta_{ux}u_x u_y - 2\tau_{xu}u_x u_t - \xi_{uu}u_x^3 \\ &\quad -3\xi_u u_x u_{xx} - \eta_{uu}u_x^2 u_y - \tau_{uu}u_x^2 u_t - 2\eta_u u_x u_{xy} - \eta_u u_{xx}u_y - \tau_u u_{xx}u_t - 2\tau_u u_x u_{xt} \ , \\ \phi^{yy} &= D_y D_y(\phi - \xi u_x - \eta u_y - \tau u_t) + \xi u_{xyy} + \eta u_{yyy} + \tau u_{tyy} \\ &= \phi_{yy} - \xi_{yy}u_x + (2\phi_{yu} - \eta_{yy})u_y - \tau_{yy}u_t - 2\xi_y u_{xy} + (\phi_u - 2\eta_y)u_{yy} \\ &\quad -2\xi_{yu}u_x u_y - 2\tau_{yu}u_y u_t + (\phi_{uu} - 2\eta_{yu})u_y^2 - 2\tau_y u_{yt} - 2\xi_u u_y u_{xy} \\ &\quad -3\eta_u u_y u_{yy} - \xi_{uu}u_y^2 u_x - \xi_u u_{yy}u_x - \eta_{uu}u_y^3 - 2\tau_u u_y u_{yt} - \tau_u u_{yy}u_t - \tau_{uu}u_y^2 u_t \ , \\ \phi^{xt} &= D_x D_t(\phi - \xi u_x - \eta u_y - \tau u_t) + \xi u_{xxt} + \eta u_{xyt} + \tau u_{xtt} \\ &= \phi_{xt} + (\phi_{tu} - \xi_{xt})u_x - \eta_{xt}u_y + (\phi_{xu} - \tau_{xt})u_t + (\phi_{uu} - \xi_{xu} - \tau_{ut})u_x u_t \\ &\quad +(\phi_u - \xi_x - \tau_t)u_{xt} - \xi_{tu}u_x^2 - \tau_{xu}u_t^2 - \xi_t u_{xx} - \xi_{uu}u_x^2 u_t - \xi_u u_{xx}u_t \\ &\quad -2\xi_u u_x u_{xt} - \eta_{tu}u_x u_y - \eta_t u_{xy} - \eta_{xu}u_y u_t - \eta_x u_{yt} - \eta_{uu}u_x u_y u_t \\ &\quad -\eta_u u_{xy}u_t - \eta_u u_y u_{xt} - \eta_u u_x u_{yt} - \tau_{uu}u_x u_t^2 - 2\tau_u u_t u_{xt} - \tau_x u_{tt} - \tau_u u_x u_{tt} \ , \\ \phi^{xxxx} &= D_x D_x D_x D_x(\phi - \xi u_x - \eta u_y - \tau u_t) + \xi u_{xxxxx} + \eta u_{yxxxx} + \tau u_{txxxx} \end{aligned}$$

$$
\begin{aligned}
\phi^{xxxx} &= \phi_{xxxx} + (4\phi_{xxxu} - \xi_{xxxx})u_x + (6\phi_{xxu} - 4\xi_{xxx})u_{xx} + (6\phi_{xxuu} - 4\xi_{xxxu})u_x^2 \\
&+(4\phi_{uuux} - 6\xi_{xxuu})u_x^3 + (\phi_{uuuu} - 4\xi_{xuuu})u_x^4 + (12\phi_{xuu} - 18\xi_{xxu})u_x u_{xx} \\
&+(6\phi_{uuu} - 24\xi_{xuu})u_x^2 u_{xx} + (3\phi_{uu} - 12\xi_{xu})u_{xx}^2 + (4\phi_{xu} - 6\xi_{xx})u_{xxx} \\
&+(4\phi_{uu} - 16\xi_{xu})u_x u_{xxx} + (\phi_u - 4\xi_x)u_{xxxx} - \xi_{uuuu}u_x^5 - 10\xi_{uuu}u_x^3 u_{xx} \\
&-15\xi_{uu}u_x u_{xx}^2 - 10\xi_{uu}u_x^2 u_{xxx} - 10\xi_u u_{xx}u_{xxx} - 5\xi_u u_x u_{xxxx} - \eta_{xxxx}u_y \\
&-\tau_{xxxx}u_t - 4\eta_{xxxu}u_x u_y - 4\eta_{xxx}u_{xy} - (12\eta_{xxu} + 12\eta_{xu})u_x u_{xy} \\
&-6\eta_{xxuu}u_x^2 u_y - 4\eta_{xuuu}u_x^3 u_y - \eta_{uuuu}u_x^4 u_y - 12\eta_{xuu}u_x^2 u_{xy} - 4\eta_{uuu}u_x^3 u_{xy} \\
&-6\eta_{xxu}u_{xx}u_y - 12\eta_{xuu}u_x u_y u_{xx} - 6\eta_{uuu}u_x^2 u_y u_{xx} - 3\eta_{uu}u_y u_{xx}^2 \\
&-12\eta_{uu}u_x u_{xy}u_{xx} - 4\eta_{ux}u_{xxx}u_y - 4\eta_{uu}u_x u_y u_{xx} - \eta_u u_{xxxx}u_y \\
&-4\eta_u u_{xxx}u_{xy} - 12\eta_{xu}u_{xx}u_{xy} - 6\eta_{xx}u_{xxy} - 6\eta_{uu}u_x^2 u_{xxy} - 6\eta_u u_{xx}u_{xxy} \\
&-4\eta_x u_{xxxy} - 4\eta_u u_x u_{xxxy} - 4\tau_{xxxu}u_x u_t - 4\tau_{xxx}u_{xt} - 12\tau_{xxu}u_x u_{xt} \\
&-6\tau_{xxuu}u_x^2 u_t - 4\tau_{xuuu}u_x^3 u_t - \tau_{uuuu}u_x^4 u_t - 12\tau_{xuu}u_x^2 u_{xt} - 4\tau_{uuu}u_x^3 u_{xt} \\
&-6\tau_{xxu}u_{xx}u_t - 12\tau_{xuu}u_x u_{xx}u_t - 6\tau_{uuu}u_x^2 u_{xx}u_t - 3\tau_{uu}u_{xx}^2 u_t \\
&-4\tau_{xu}u_{xxx}u_t - 4\tau_{uu}u_x u_{xxx}u_t - \tau_u u_{xxxx}u_t - 4\tau_u u_{xxx}u_{xt} - 12\tau_{xu}u_{xx}u_{xt} \\
&-12\tau_{uu}u_x u_{xx}u_{xt} - 6\tau_{xx}u_{xxt} - 12\tau_{xu}u_x u_{xt} - 6\tau_{uu}u_x^2 u_{xxt} - 6\tau_u u_{xx}u_{xxt} \\
&-4\tau_x u_{xxxt} - 4\tau_u u_x u_{xxxt}\ .
\end{aligned}
$$

Substitute them in Equation (12) and then compare coefficients of various monomials in derivatives of 'u'. This yields the following equations:

$$
\begin{aligned}
\xi_u &= 0\ , \\
\eta_u &= 0\ , \\
\tau_u &= 0\ , \\
\phi_{uu} &= 0\ , \\
\tau_y &= 0\ , \\
\eta_x &= 0\ , \\
\tau_x &= 0\ , \\
\phi_{xu} &= 0\ , \\
\xi_{xx} &= 0\ , \\
0 &= \phi_x + \phi_{xt} + u^3\phi_{xx} + \beta\phi_{yy} + \alpha\phi_{xxxx}\ , \\
0 &= \tau_t + 6u^2\phi_x - \xi_{xt} - \beta\xi_{yy} + \phi_{tu} + 4\alpha\phi_{xxxu}\ , \\
0 &= 3u^2\phi - \xi_t - u^3\xi_x + u^3\tau_t + 6\alpha\phi_{xxu}\ , \\
0 &= 2\phi - u\xi_x + u\tau_t + u\phi_u\ , \\
0 &= \xi_x - 2\eta_y + \tau_t\ , \\
\tau_t &= 3\xi_x\ , \\
\eta_{yy} &= 2\phi_{yu}\ , \\
\eta_t &= -2\beta\xi_y\ .
\end{aligned}
$$

After some simplifications, we get, the following PDEs,

$$\begin{aligned}
\tau &= \tau(t) \ , && (17)\\
\eta_x &= \xi_u \ = \ \eta_u \ = \ 0 \ , && (18)\\
\xi &= g(y,t)x + h(y,t) \ , && (19)\\
\phi_{xu} &= \phi_{uu} \ = \ 0 \ , && (20)\\
0 &= \phi_x + \phi_{xt} + u^3\phi_{xx} + \beta\phi_{yy} + \alpha\phi_{xxxx} \ , && (21)\\
0 &= \tau_t + 6u^2\phi_x - \xi_{xt} - \beta\xi_{yy} + \phi_{tu} + 4\alpha\phi_{xxxu} \ , && (22)\\
0 &= 3u^2\phi - \xi_t - u^3\xi_x + u^3\tau_t + 6\alpha\phi_{xxu} \ , && (23)\\
0 &= 2\phi - u\xi_x + u\tau_t + u\phi_u \ , && (24)\\
0 &= \xi_x - 2\eta_y + \tau_t \ , && (25)\\
\tau_t &= 3\xi_x \ , && (26)\\
\eta_{yy} &= 2\phi_{yu} \ , && (27)\\
\eta_t &= -2\beta\xi_y \ . && (28)
\end{aligned}$$

Using the above equations and some more manipulations, we get,

$$\begin{aligned}
\xi &= k_3 + k_4 y \ , && (29)\\
\eta &= k_1 - 2k_4 t\beta \ , && (30)\\
\tau &= k_2 \ , && (31)\\
\phi &= 0 \ . && (32)
\end{aligned}$$

At this stage, we construct the symmetry generators corresponding to each of the constants involved. These are a total of four generators given by

$$\begin{aligned}
V_1 &= \frac{\partial}{\partial y} \ ,\\
V_2 &= \frac{\partial}{\partial t} \ ,\\
V_3 &= \frac{\partial}{\partial x} \ ,\\
V_4 &= y\frac{\partial}{\partial x} - 2t\beta\frac{\partial}{\partial y} \ . && (33)
\end{aligned}$$

The one-parameter groups $g_i(\epsilon)$ generalized by the V_i, where i=1, 2, 3, 4, are

$$\begin{aligned}
g_1(\epsilon) : (x,y,t;u) &\rightarrow (x, y+\epsilon, t, u) \ ,\\
g_2(\epsilon) : (x,y,t;u) &\rightarrow (x, y, t+\epsilon, u) \ ,\\
g_3(\epsilon) : (x,y,t;u) &\rightarrow (x+\epsilon, y, t, u) \ ,\\
g_4(\epsilon) : (x,y,t;u) &\rightarrow (x+y\epsilon, y-2t\beta\epsilon, t, u) \ ,
\end{aligned}$$

where $exp(\epsilon V_i)\ (x,y,t;u) = (\bar{x},\bar{y},\bar{t};\bar{u})$ and

(i) g_2 is time translation ,

(ii) g_1, g_3 and g_4 are the space-invariant of the equation. The symmetry generators found in Equation (33) form a closed Lie Algebra whose commutation table is shown below.

$[V_i, V_j]$	V_1	V_2	V_3	V_4
V_1	0	0	0	V_3
V_2	0	0	0	$-2\beta V_1$
V_3	0	0	0	0
V_4	$-V_3$	$2\beta V_1$	0	0

Table 1. **Commutation relations satisfied by above generators is**

The commutation relations of the Lie algebra, determined by V_1, V_2, V_3 and V_4 are shown in the above table. These vector fields form a Lie algebra L by:

$$[V_1, V_4] = V_3 \ , \ [V_2, V_4] = \ -2\beta V_1 \ .$$

For this four-dimensional Lie algebra the commutator table for V_i is a $(4 \otimes 4)$ table whose $(i, j)^{th}$ entry expresses the Lie Bracket $[V_i, V_j]$ given by the above Lie algebra L. The table is skew-symmetric and the diagonal elements all vanish. The coefficient $C_{i,j,k}$ is the coefficient of V_i of the $(i, j)^{th}$ entry of the commutator table and the related structure constants can be easily calculated from above table are as follows:

$$C_{1,4,3} = 1 \ , \ C_{2,4,1} = -2\beta \ .$$

The Lie algebra L is solvable. The radical of G is ,

$$R =< V_3 > \oplus < V_1, V_2, V_4 >.$$

In the next section, we derive the reduction of (6) to PDEs with two independent variables and ODEs. These are four one-dimensional Lie subalgebras

$$L_{s,1} = \{V_1\} \ , \ L_{s,2} = \{V_2\} \ , \ L_{s,3} = \{V_3\} \ , \ L_{s,4} = \{V_4\}$$

and corresponding to each one-dimensional subalgebras we may reduce (6) to a PDE with two independent variables. Further reductions to ODEs are associated with two-dimensional subalgebras. It is evident from the commutator table that there are no two-dimensional solvable non-abelian subalgebras. And there are four two-dimensional Abelian subalgebras, namely,

$$L_{A,1} = \{V_1, V_2\} \ , \ L_{A,2} = \{V_1, V_3\} \ , \ L_{A,3} = \{V_2, V_3\} \ , \ L_{A,4} = \{V_3, V_4\} \ .$$

3. Reductions of $(u_t + u + u^3 u_x + \alpha u_{xxx})_x + \beta u_{yy} = 0$ by One-Dimensional Subalgebras

Case 1 : $V_1 = \partial_y$.

The characteristic equation associated with this generator is

$$\frac{dx}{0} = \frac{dy}{1} = \frac{dt}{0} = \frac{du}{0} \ .$$

We integrate the characteristic equation to get three similarity variables,

$$x = s, \quad t = r \quad \text{and} \quad u = w(r, s) \ . \tag{34}$$

Using these similarity variables in Equation (6) can be recast in the form

$$w_s + 3w^2 w_s^2 + w_{sr} + w^3 w_{ss} + \alpha w_{ssss} = 0 \ . \tag{35}$$

Case 2 : $V_2 = \partial_t$.

The characteristic equation associated with this generator is

$$\frac{dx}{0} = \frac{dy}{0} = \frac{dt}{1} = \frac{du}{0} \ .$$

Following standard procedure we integrate the characteristic equation to get three similarity variables,

$$x = s, \quad y = r \quad \text{and} \quad u = w(r,s) \ . \tag{36}$$

Using these similarity variables in Equation (6) can be recast in the form

$$w_s + 3w^2 w_s^2 + w^3 w_{ss} + \beta w_{rr} + \alpha w_{ssss} = 0 \ . \tag{37}$$

Case 3 : $V_3 = \partial_x$.

The characteristic equation associated with this generator is

$$\frac{dx}{1} = \frac{dy}{0} = \frac{dt}{0} = \frac{du}{0} \ .$$

Following standard procedure we integrate the characteristic equation to get three similarity variables,

$$y = s, \quad t = r \quad \text{and} \quad u = w(r,s) \ . \tag{38}$$

Using these similarity variables in Equation (6) can be recast in the form

$$\beta w_{ss} = 0 \ . \tag{39}$$

Case 4 : $V_4 = y\partial_x - 2t\beta\partial_y$.

The characteristic equation associated with this generator is

$$\frac{dx}{y} = \frac{dy}{-2t\beta} = \frac{dt}{0} = \frac{du}{0} \ .$$

Following standard procedure we integrate the characteristic equation to get three similarity variables,

$$s = -(y^2 + 4r\beta x), \quad t = r \quad \text{and} \quad u = w(r,s) \ . \tag{40}$$

Using these similarity variables in Equation (6) can be recast in the form

$$\begin{aligned} 4r\beta w_s &= 48w^2 w_{ss} r^2 \beta^2 - 4w_s r\beta' - 4s\beta w_{ss} + 16w^3 w_{ss} r^2 \beta^2 - 6\beta w_s \\ &\quad + 256 r^4 \beta^4 \alpha w_{ssss} \ . \end{aligned} \tag{41}$$

4. Reductions of $(u_t + u + u^3 u_x + \alpha u_{xxx})_x + \beta u_{yy} = 0$ by Two-Dimensional Subalgebras

Case I : Reduction under V_1 and V_2 .

From Table 1 we find that the given generators commute $[V_1, V_2] = 0$. Thus either of V_1 or V_2 can be used to start the reduction with. For our purpose we begin reduction with V_1. Therefore we get Equation (34) and Equation (35). At this stage, we express V_2 in terms of the similarity variables defined in (34). The transformed V_2 is

$$\tilde{V_2} = \partial_r \ .$$

The characteristic equation for $\tilde{V_2}$ is

$$\frac{ds}{0} = \frac{dr}{1} = \frac{dw}{0} \ .$$

Integrating this equation as before leads to new variables

$$s = \gamma \ \text{ and } \ w = k(\gamma),$$

which reduce Equation (35) to a fourth-order ODE

$$k' + 3k^2 k'^2 + k^3 k'' + \alpha k'''' = 0 \ . \tag{42}$$

Case II : Reduction under V_1 and V_3 .

From Table 1 we find that the given generators commute $[V_1, V_3] = 0$. Thus either of V_1 or V_3 can be used to start the reduction with. For our convenience we begin reduction with V_3. Therefore we get Equation (38) and Equation (39). At this stage, we express V_1 in terms of the similarity variables defined in Equation (38). The transformed V_1 is

$$\tilde{V_1} = \partial_s \ .$$

The characteristic equation for $\tilde{V_1}$ is

$$\frac{ds}{1} = \frac{dr}{0} = \frac{dw}{0} \ .$$

Integrating this equation as before leads to new variables

$$r = \gamma \ \text{ and } \ w = k(\gamma) \ .$$

It follows that Equation (39) is satisfied. So, now reduction start with V_1. Therefore we get Equation (34) and Equation (35). At this stage, we express V_2 in terms of the similarity variables defined in Equation (34). The transformed V_3 is

$$\tilde{V_3} = \partial_s \ .$$

Similarly of above procedure, Equation (35) is satisfied.

Case III : Reduction under V_2 and V_3 .

In this case the two symmetry generators V_2 and V_3 satisfy the commutation relation $[V_2, V_3] = 0$. This suggests that reduction in this case should start with V_3. The similarity variables are

$$y = s, \ t = r \ \text{ and } \ u = w(r,s).$$

The corresponding reduced PDE is

$$\beta w_{ss} = 0 \ .$$

The transformed V_2 is

$$\tilde{V_2} = \partial_r \ .$$

The invariants of $\tilde{V_2}$ are

$$s = \gamma \ \text{ and } \ w = k(\gamma) \ ,$$

which reduce Equation (39) to the ODE

$$\beta k'' = 0 \ . \tag{43}$$

Case IV : Reduction under V_3 and V_4 .

In this case the two symmetry generators V_3 and V_4 satisfy the commutation relation $[V_3, V_4] = 0$. This suggests that reduction in this case should start with V_3. Therefore we get Equation (38) and Equation (39). The transformed V_4 is

$$\tilde{V_4} = -2r\beta\partial_s \ .$$

The invariants of $\tilde{V_4}$ are

$$r = \gamma \ \text{ and } \ w = k(\gamma) \ .$$

It follows that Equation (39) is satisfied. So, now reduction start with V_4. Therefore we get Equation (40) and Equation (41). Now transforming V_3 in these new variables is given by

$$\tilde{V_3} = -4r\beta\partial_s \ .$$

The invariants of $\tilde{V_3}$ are

$$r = \gamma \ \text{ and } \ w = k(\gamma) \ .$$

In these variables Equation (41) is satisfied.

Algebra	Reduction
$[V_1, V_2] = 0$	$k' + 3k^2k'^2 + k^3k'' + \alpha k'''' = 0$
$[V_1, V_3] = 0$	Satisfy the Equation
$[V_2, V_3] = 0$	$\beta k'' = 0$
$[V_3, V_4] = 0$	Satisfy the Equation

5. Conclusions

In this Paper,

(1). A $(2+1)$-dimensional KdV equation $(u_t + u + u^3 u_x + \alpha u_{xxx})_x + \beta u_{yy} = 0$ where $\alpha, \ \beta \in \mathbb{R}$, is subjected to Lie's classical method.

(2). Equation (6) admits a four-dimensional symmetry group.

(3). It is established that the symmetry generators form a closed Lie algebra.

(4). Classification of symmetry algebra of (6) into one- and two-dimensional subalgebras is carried out.

(5). Systematic reduction to (1+1)-dimensional PDE and then to first- or second order ODEs are performed using one-dimensional and two-dimensional solvable Abelian subalgebras.

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