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Weakly $(\sigma_{\mathcal{H}}, \lambda)$-continuity in Generalized Topologies

Research Article

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Abstract: In this paper, we introduce and study the notions of weakly σ-$\mathcal{H}$-open sets and weakly $(\sigma_{\mathcal{H}}, \lambda)$-continuity in a hereditary generalized topological space. Also we prove that a function $f : (X, \mu, \mathcal{H}) \to (Y, \lambda)$ is (μ, λ)-continuous if and only if f is weakly $(\sigma_{\mathcal{H}}, \lambda)$-continuous and strong $(S_{\mathcal{H}}, \lambda)$-continuous.

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1. Introduction and Preliminaries

A family μ of subset of X is called a generalized topology(GT)[1] if $\emptyset \in \mu$ and closed under arbitrary union. The generalized topology μ is said to be strong [10], if $X \in \mu$. (X, μ) is called a quasi topology [6], if μ is closed under finite intersection. A subset A of a generalized topological space (X, μ) is called μ-σ-open [3] (resp. μ-π-open [3], μ-α-open [3], μ-β-open [3]) if $A \subset c_\mu(i_\mu(A))$ (resp. $A \subset i_\mu(c_\mu(A))$ $A \subset i_\mu(c_\mu(i_\mu(A)))$, $A \subset c_\mu(i_\mu(c_\mu(A)))$. c_σ is the intersection of all μ-σ-closed set containing A. A function $f : (X, \mu, \mathcal{H}) \to (Y, \lambda)$ is said to be (μ, λ)-continuous if for every λ-open set U in Y implies that $f^{-1}(U)$ is μ-open set in X. A hereditary class $\mathcal{H}$ of X is non-empty collection of subset of X such that $A \subset B$, $B \in \mathcal{H}$ implies $A \in \mathcal{H}$ [2]. In the paper [2], for a hereditary class $\mathcal{H}$, the operator $()^*$:exp X$\to$ exp X was introduced. An operator c^*_μ : exp X $\to$ exp X was defined by using the operator $()^*$ (i.e., for $A \subset X$) $c^*_\mu(A) = A \cup A^*$, which is monotonic, enlarging and idempotent. Some properties of operators $()^*$ and c^*_μ were investigated in [2]. For every subset A of X, with respect to μ and a hereditary class $\mathcal{H}$ of subset of X, then $\mu^* = \{A \subset X / c^*_\mu(X - A) = X - A\}$ is generalized topology[2], and $i^*_\mu(A)$ will denote the interior of A in (X, μ^*). A subset A of a hereditary generalized topological space $(X, \mu, \mathcal{H})$ is said to be α-$\mathcal{H}$-open [2](resp. β-$\mathcal{H}$-open [2], σ-$\mathcal{H}$-open, π-$\mathcal{H}$-open [2], δ-$\mathcal{H}$-open [2], t-$\mathcal{H}$-set [7], t^*-$\mathcal{H}$-set [7]) if $A \subseteq i_\mu(c^*_\mu(i_\mu(A)))$ (resp. $A \subseteq c_\mu(i_\mu(c^*_\mu(A)))$, $A \subseteq c^*_\mu(i_\mu(A))$, $A \subseteq i_\mu(c^*_\mu(A))$, $i_\mu(c^*_\mu(A)) \subseteq c^*_\mu(i_\mu(A))$, $i_\mu(c^*_\mu) = i_\mu(A)$, $i_\mu(c^*_\mu(i_\mu(A))) = i_\mu(A)$). If A is μ^*-closed [2] if $A^* \subset A$.

Lemma 1.1 ([2]). *If (X, μ) is GTS with a hereditary class $\mathcal{H}$. If A and B are any two subets of X, then the following hold.*

(a) If $A \subset B$, then $A^* \subset B^*$.

(b) $A^* = c^*_\mu(A) \subset c^*_\mu(A^*)$.

(c) If $A \subset A^*$, then $c_\mu(A) = A^* = c^*_\mu(A) = c^*_\mu(A^*)$.

(d) If $U \in \mu$, then $U \cap A^* \subset (U \cap A)^*$.

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Lemma 1.2 ([7]). *If (X, μ) is quasi topology with a hereditary class $\mathcal{H}$. Then the following hold.*

(a) $\mathcal{H}$ is μ-codense if and only if $A \subset A^*$ for every $A \in \mu$.

(b) If $A \subset A^*$, then $A^* = c_\mu(A^*) = c_\mu(A) = c^*_\mu(A)$.

Lemma 1.3 ([9]). *If (X, μ) is GTS with a hereditary class $\mathcal{H}$. For $A \subset X$,*

(a) $c^*_\mu(A) = X$-$i^*_\mu(X$-$A)$.

(b) $i_\mu(A) \subset i^*_\mu(A) \subset A$.

2. Weakly σ-$\mathcal{H}$-open Set

Definition 2.1. *A subset A of a hereditary generalized topological space $(X, \mu, \mathcal{H})$ is said to be weakly σ-$\mathcal{H}$-open, if $A \subset c^*_\mu(i_\mu(c_\mu(A)))$. A subset A of X is said to be weakly σ-$\mathcal{H}$-closed if its complement is weakly σ-$\mathcal{H}$-open.*

Proposition 2.2. *In a hereditary generalized topological space $(X, \mu, \mathcal{H})$, the following hold.*

(a) Every α-$\mathcal{H}$-open is weakly σ-$\mathcal{H}$-open.

(b) Every strong β-$\mathcal{H}$-open is weakly σ-$\mathcal{H}$-open.

(c) Every σ-$\mathcal{H}$-open is weakly σ-$\mathcal{H}$-open.

(d) Every π-$\mathcal{H}$-open is weakly σ-$\mathcal{H}$-open.

Theorem 2.3. *Let (X, μ) be a quasi topology with a hereditary class $\mathcal{H}$ and $\mathcal{H}$ be a μ-codense and $A \subset X$. If A is σ-$\mathcal{H}$-open if and only if it is both weakly σ-$\mathcal{H}$-open and δ-$\mathcal{H}$-open.*

Proof. Necessity: By Proposition 2.2, A is weakly σ-$\mathcal{H}$ open. Now we prove that $i_\mu(c^*_\mu(A)) \subset c^*_\mu(i_\mu(A))$. Since A is σ-$\mathcal{H}$-open, implies that $A \subset c^*_\mu(i_\mu(A))$, so, $i_\mu(c^*_\mu(A)) \subset i_\mu(c^*_\mu(c^*_\mu(i_\mu(A)))) \subset i_\mu(c^*_\mu(i_\mu(A))) \subset c^*_\mu(i_\mu(A))$. Hence A is δ-$\mathcal{H}$-open.
Sufficiency: Let A be both δ-$\mathcal{H}$-open and weakly σ-$\mathcal{H}$-open. Then we have $A \subset c^*_\mu(i_\mu(c_\mu(A))) = c^*_\mu(i_\mu(c^*_\mu(A))) \subset c^*_\mu(c^*_\mu(i_\mu(A))) \subset c^*_\mu(i_\mu(A))$. Hence A is δ-$\mathcal{H}$-open. □

Theorem 2.4. *Let $(X, \mu, \mathcal{H})$ be a hereditary generalized topological space. Then any arbitrary union of weakly σ-$\mathcal{H}$-open sets is weakly σ-$\mathcal{H}$-open.*

Proof. Let U_α be weakly σ-$\mathcal{H}$-open for every $\alpha \in \Delta$, we have $U_\alpha \subset c^*_\mu(i_\mu(c_\mu(U_\alpha)))$ for every $\alpha \in \Delta$. Then $\cup_{\alpha\in\Delta} U_\alpha \subset \cup_{\alpha\in\Delta} c^*_\mu(i_\mu(c_\mu(U_\alpha))) = \cup_{\alpha\in\Delta}((i_\mu(c_\mu(U_\alpha)))^* \cup i_\mu(c_\mu(U_\alpha))) \subset (\cup_{\alpha\in\Delta} i_\mu(c_\mu(U_\alpha)))^* \cup (i_\mu(c_\mu(\cup_{\alpha\in\Delta} U_\alpha))) \subset (i_\mu(c_\mu(\cup_{\alpha\in\Delta})))^* \cup (i_\mu(c_\mu(\cup_{\alpha\in\Delta} U_\alpha))) = c^*_\mu(i_\mu(c_\mu(\cup_{\alpha\in\Delta} U_\alpha)))$. Hence $\cup_{\alpha\in\Delta} U_\alpha$ is weakly σ-$\mathcal{H}$-open. □

Remark 2.5. *The following example shows that the intersection of weakly σ-$\mathcal{H}$-open sets need not be a weakly σ-$\mathcal{H}$-open set.*

Example 2.6. *Consider a hereditary generalized topological space $(X, \mu, \mathcal{H})$ where $X = \{a, b, c, d\}$, $\mu = \{\emptyset, \{b\}, \{b, c\}, \{c, d\}, \{b, c, d\}\}$ and $\mathcal{H} = \{\emptyset, \{a\}, \{b\}\}$. If $A = \{b, d\}$ and $B = \{c, d\}$, then $c^*_\mu(i_\mu(c_\mu(A))) = c^*_\mu(i_\mu(X)) = c^*_\mu(\{b, c, d\}) = X \supset A$ and $c^*_\mu(i_\mu(c_\mu(B))) = c^*_\mu(i_\mu(\{a, c, d\})) = c^*_\mu(\{c, d\}) = \{a, c, d\} \supset B$. Hence A and B are weakly σ-$\mathcal{H}$-open sets. But $c^*_\mu(i_\mu(c_\mu(A \cap B))) = c^*_\mu(i_\mu(c_\mu(\{d\}))) = c^*_\mu(i_\mu(\{a, d\})) = c^*_\mu(\emptyset) = \emptyset \not\supseteq A \cap B$, so, $A \cap B$ is not weakly σ-$\mathcal{H}$-open.*

Theorem 2.7. *Let $(X, \mu, \mathcal{H})$ be a hereditary generalized topological space and A, $B \subset X$. If A is weakly σ-$\mathcal{H}$-open and $B \in \mu$, then $A \cap B$ is weakly σ-$\mathcal{H}$-open.*

Proof. If A is weakly σ-$\mathcal{H}$-open implies that $A \subset c_\mu^*(i_\mu(c_\mu(A)))$ and $i_\mu(B) = B$. Then $A \cap B \subset c_\mu^*(i_\mu(c_\mu(A))) \cap B = ((i_\mu(c_\mu(A)))^* \cup i_\mu(c_\mu(A))) \cap B = ((i_\mu(c_\mu(A)))^* \cap B) \cup (i_\mu(c_\mu(A)) \cap B) \subset (i_\mu(c_\mu(A)) \cap B)^* \cup (i_\mu(c_\mu(A)) \cap B) = (i_\mu(c_\mu(A)) \cap i_\mu(B))^* \cup (i_\mu(c_\mu(A)) \cap i_\mu(B)) = (i_\mu(c_\mu(A) \cap B))^* \cup (i_\mu(c_\mu(A) \cap B)) \subset (i_\mu(c_\mu(A \cap B))^* \cup (i_\mu(c_\mu(A \cap B)) = c_\mu^*(i_\mu(c_\mu(A \cap B)))$. Hence $A \cap B$ is weakly σ-$\mathcal{H}$-open. □

Remark 2.8. *The following theorem establish that, in the above result, μ-openness can be replaced by α-$\mathcal{H}$-openness.*

Theorem 2.9. *Let $(X, \mu, \mathcal{H})$ be a hereditary generalized topological space and A, $B \subset X$. If A is weakly σ-$\mathcal{H}$-open and B is α-$\mathcal{H}$-open, then $A \cap B$ is weakly σ-$\mathcal{H}$-open.*

Proof. Since A is α-$\mathcal{H}$-open implies $A \subset i_\mu(c_\mu^*(i_\mu(A)))$ and B is weakly σ-$\mathcal{H}$-open implies $B \subset c_\mu^*(i_\mu(c_\mu(B)))$. Then $A \cap B \subset c_\mu^*(i_\mu(c_\mu(A))) \cap i_\mu(c_\mu^*(i_\mu(B))) = ((i_\mu(c_\mu(A)))^* \cup (i_\mu(c_\mu(A)))) \cap i_\mu(c_\mu^*(i_\mu(B))) = ((i_\mu(c_\mu(A)))^* \cap i_\mu(c_\mu^*(i_\mu(B)))) \cup (i_\mu(c_\mu(A)) \cap i_\mu(c_\mu^*(i_\mu(B)))) \subset (i_\mu(c_\mu(A)) \cap i_\mu(c_\mu^*(i_\mu(B))))^* \cup i_\mu(i_\mu(c_\mu(A)) \cap c_\mu^*(i_\mu(B))) = (i_\mu(i_\mu(c_\mu(A)) \cap c_\mu^*(i_\mu(B)))^* \cup i_\mu(i_\mu(c_\mu(A)) \cap c_\mu^*(i_\mu(B))) \subset (i_\mu(c_\mu^*(i_\mu(c_\mu(A)) \cap i_\mu(B))))^* \cup i_\mu(c_\mu^*(i_\mu(c_\mu(A)) \cap i_\mu(B))) = (i_\mu(c_\mu^*(i_\mu(c_\mu(A) \cap i_\mu(B)))))^* \cup i_\mu(c_\mu^*(i_\mu(c_\mu(A) \cap i_\mu(B)))) \subset (i_\mu(c_\mu^*(i_\mu(c_\mu(A \cap i_\mu(B))))))^* \cup i_\mu(c_\mu^*(i_\mu(c_\mu(A \cap i_\mu(B))))) \subset (i_\mu(c_\mu^*(i_\mu(c_\mu(A \cap B)))))^* \cup i_\mu(c_\mu^*(i_\mu(c_\mu(A \cap B)))) \subset (i_\mu(c_\mu(i_\mu(c_\mu(A \cap B)))))^* \cup i_\mu(c_\mu(i_\mu(c_\mu(A \cap B)))) = (i_\mu(c_\mu(A \cap B)))^* \cup i_\mu(c_\mu(A \cap B)) = c_\mu^*(i_\mu(c_\mu(A \cap B)))$. Hence $A \cap B$ is weakly σ-$\mathcal{H}$-open. □

Proposition 2.10. *Let $(X, \mu, \mathcal{H})$ be a hereditary generalized topological space and A, $B \subset X$. Then*

(a) If $A \subset B \subset c_\mu^*(A)$ and A is weakly σ-$\mathcal{H}$-open, then B, A^* and B^* are weakly σ-$\mathcal{H}$-open sets.

(b) If $A \subset B \subset c_\mu^*(A)$ and A is π-$\mathcal{H}$-open, then B is strong β-$\mathcal{H}$-open.

(c) If $A \subset B \subset c_\mu(A)$ and A is π-$\mathcal{H}$-open, then B is β-$\mathcal{H}$-open.

Proof. (a) Suppose that $A \subset B \subset c_\mu^*(A)$ and A is weakly σ-$\mathcal{H}$-open implies that $A \subset c_\mu^*(i_\mu(c_\mu(A)))$. Since $B \subset c_\mu^*(A) \subset c_\mu^*(i_\mu(c_\mu(A))) \subset c_\mu^*(i_\mu(c_\mu(B)))$. Hence B is weakly σ-$\mathcal{H}$-open. Since $A \subset B \subset A^*$, we have B,A^* and B^* are weakly σ-$\mathcal{H}$-open sets.

(b) Suppose $A \subset B \subset c_\mu^*(A)$ and A is π-$\mathcal{H}$-open implies that $A \subset i_\mu(c_\mu^*(A))$. Now $B \subset c_\mu^*(A) \subset c_\mu^*(i_\mu(c_\mu^*(A))) \subset c_\mu^*(i_\mu(c_\mu^*(B)))$. Hence B is strong β-$\mathcal{H}$-open.

(c) Suppose $A \subset B \subset c_\mu(A)$ and A is π-$\mathcal{H}$-open implies that $A \subset i_\mu(c_\mu^*(A))$. Now $B \subset c_\mu(A) \subset c_\mu(i_\mu(c_\mu^*(A))) \subset c_\mu(i_\mu(c_\mu^*(B)))$. Hence B is β-$\mathcal{H}$-open. □

Corollary 2.11. *Let $(X, \mu, \mathcal{H})$ be a hereditary generalized topological space and $A \subset X$. Then the following hold.*

(a) If A is weakly σ-$\mathcal{H}$-open, then $c_\mu^*(A)$ and $c_\mu^*(i_\mu(c_\mu^*(A)))$ are weakly σ-$\mathcal{H}$-open sets.

(b) If A is π-$\mathcal{H}$-open, then $c_\mu^*(A)$ and $c_\mu^*(i_\mu(c_\mu^*(A)))$ are strong β-$\mathcal{H}$-open sets.

Corollary 2.12. *Let $(X, \mu, \mathcal{H})$ be a hereditary generalized topological space and $A \subset X$ be a weakly σ-$\mathcal{H}$-open. Then the following hold.*

(a) If A is $A \subset A^*$, then A^* is weakly σ-$\mathcal{H}$-open.

(b) If A is $A = A^*$, then every subset containing A is strong β-$\mathcal{H}$-open.

Theorem 2.13. *Let $(X, \mu, \mathcal{H})$ be a hereditary generalized topological space. If $\mu = \{\emptyset\}$ (resp. $P(X)$, $\mathcal{N}$), then the set of all weakly σ-$\mathcal{H}$-open sets is same as the set of all μ-β-open sets (resp. the set of all weakly σ-$\mathcal{H}$-open sets is same as the set of all μ-π-open sets, the set of all weakly σ-$\mathcal{H}$-open sets is same as the set of all μ-β-open sets).*

Proof. (i) If $\mathcal{H} = \{\emptyset\}$ then $A^* = c_\mu(A)$ and hence $c_\mu^*(A) = A \cup A^* = c_\mu(A)$ for every subset A of $(X, \mu, \mathcal{H})$. Therefore, $c_\mu^*(i_\mu(c_\mu(A))) = c_\mu(i_\mu(c_\mu(A)))$.

(ii) Let $\mathcal{H} = P(X)$, then $A^* = \emptyset$ and $c^*_\mu(A) = A$ for every subset A of X. Therefore, $c^*_\mu(i_\mu(c_\mu(A))) = i_\mu(c_\mu(A))$.

(iii) Let $\mathcal{H} = \mathcal{N}$, then $A^* = c_\mu(i_\mu(c_\mu(A)))$ for every subset A of X. Therefore, we have $c^*_\mu(i_\mu(c_\mu(A))) = (i_\mu(c_\mu(A)))^* \cup i_\mu(c_\mu(A)) = c_\mu(i_\mu(c_\mu(i_\mu(c_\mu(A))))) \cup i_\mu(c_\mu(A)) = c_\mu(i_\mu(c_\mu(A))) \cup i_\mu(c_\mu(A)) = c_\mu(i_\mu(c_\mu(A)))$. Hence $c^*_\mu(i_\mu(c_\mu(A))) = c_\mu(i_\mu(c_\mu(A)))$. □

Definition 2.14. *A subset A of a hereditary generalized topological space $(X, \mu, \mathcal{H})$ is said to be a strong σ-$\mathcal{H}$-open if $A \subset c^*_\mu(i_\mu(A^*))$.*

Proposition 2.15. *Let $(X, \mu, \mathcal{H})$ be a hereditary generalized topological space and $A \subset X$. Then the following are equivalent.*

(a) A is strong σ-$\mathcal{H}$-open.

(b) A is both strong β-$\mathcal{H}$-open and strong σ-$\mathcal{H}$-open.

(c) A is both weakly σ-$\mathcal{H}$-open and strong σ-$\mathcal{H}$-open.

(d) A is both weakly σ-$\mathcal{H}$-open and $A \subset A^*$.

Corollary 2.16. *Let $(X, \mu, \mathcal{H})$ be a hereditary generalized topological space where $\mathcal{H}$ is μ-codense and $A \subset X$. If $A \subset A^*$, then the following are equivalent.*

(a) A is strong σ-$\mathcal{H}$-open.

(b) A is strong β-$\mathcal{H}$-open.

(c) A is β-$\mathcal{H}$-open.

(d) A is μ-β-open.

(e) A is weakly σ-$\mathcal{H}$-open.

Theorem 2.17. *Let $(X, \mu, \mathcal{H})$ be a hereditary generalized topological space and A, $B \subset X$. If A is strong σ-$\mathcal{H}$-open and B is α-$\mathcal{H}$-open, then $A \cap B$ is σ-$\mathcal{H}$-open.*

Proof. A is strong σ-$\mathcal{H}$-open implies that $A \subset c^*_\mu(i_\mu(A^*))$ and B is α-$\mathcal{H}$-open implies that $B \subset i_\mu(c^*_\mu(i_\mu(B)))$. Now, $A \cap B \subset c^*_\mu(i_\mu(A^*)) \cap i_\mu(c^*_\mu(i_\mu(B))) = ((i_\mu(A^*))^* \cup i_\mu(A^*)) \cap i_\mu(c^*_\mu(i_\mu(B))) = ((i_\mu(A^*))^* \cap i_\mu(c^*_\mu(i_\mu(B)))) \cup (i_\mu(A^*) \cap i_\mu(c^*_\mu(i_\mu(B)))) \subset (i_\mu(A^*) \cap i_\mu(c^*_\mu(i_\mu(B))))^* \cup i_\mu(i_\mu(A^*) \cap c^*_\mu(i_\mu(B))) \subset (i_\mu(i_\mu(A^*) \cap c^*_\mu(i_\mu(B))))^* \cup i_\mu(c^*_\mu(i_\mu(A^*) \cap i_\mu(B))) \subset (i_\mu(c^*_\mu(i_\mu(A^*) \cap i_\mu(B))))^* \cup i_\mu(c^*_\mu(i_\mu(A^* \cap i_\mu(B)))) \subset (i_\mu(c^*_\mu(i_\mu(A^* \cap i_\mu(B)))))^* \cup i_\mu(c^*_\mu(i_\mu(A \cap i_\mu(B))^*)) \subset (i_\mu(c^*_\mu(i_\mu(A \cap i_\mu(B))^*)))^* \cup i_\mu(c^*_\mu(i_\mu(A \cap i_\mu(B))^*)) = c^*_\mu(i_\mu(c^*_\mu(i_\mu((A \cap i_\mu(B))^*)))) \subset c^*_\mu(i_\mu(c^*_\mu((A \cap i_\mu(B))^*))) = c^*_\mu(i_\mu((A \cap i_\mu(B))^*)) \subset c^*_\mu(i_\mu((A \cap B)^*))$. Hence $A \cap B$ is strong σ-$\mathcal{H}$-open. □

Proposition 2.18. *Let $(X, \mu, \mathcal{H})$ be a hereditary generalized topological space and $A \subset X$. If A is weakly σ-$\mathcal{H}$-open, then A is μ-β-open.*

Example 2.19. *Consider a hereditary generalized topological space $(X, \mu, \mathcal{H})$ where $X = \{a, b, c, d\}$, $\mu = \{\emptyset, \{c\}, \{b, d\}, \{a, b, c\}, \{b, c, d\}, X\}$ and $\mathcal{H} = \{\emptyset, \{d\}\}$. If $A = \{a, b\}$, then $c_\mu(i_\mu(c_\mu(A))) = c_\mu(i_\mu(\{a, b, d\})) = c_\mu(\{b, d\}) = \{a, b, d\} \supset A$. Hence A is weakly μ-β-open. But $c^*_\mu(i_\mu(c_\mu(A))) = c^*_\mu(i_\mu(\{a, b, d\})) = c^*_\mu(\{b, d\}) = \{b, d\} \not\supseteq A$, so, A is not weakly σ-$\mathcal{H}$-open.*

Theorem 2.20. *Let $(X, \mu, \mathcal{H})$ be a quasi topology with hereditary class $\mathcal{H}$ and $\mathcal{H}$ is μ-codense and $A \subset X$. Then A is μ-β-open if and only if A is weakly σ-$\mathcal{H}$-open.*

Proof. By Proposition 2.18, every weakly σ-$\mathcal{H}$-open is μ-β-open. Conversely, if A is μ-β-open implies that $A \subset c_\mu(i_\mu(c_\mu(A)))$. By Lemmas 1.2, $c_\mu(i_\mu(c_\mu(A))) = c^*_\mu(i_\mu(c_\mu(A)))$ and so $A \subset c^*_\mu(i_\mu(c_\mu(A)))$. Hence A is weakly σ-$\mathcal{H}$-open. □

Theorem 2.21. *Let $(X, \mu, \mathcal{H})$ be a hereditary generalized topological space and $A \subset X$. Then A is weakly σ-$\mathcal{H}$-closed set if and only if $i^*_\mu(c_\mu(i_\mu(A))) \subset A$.*

Proof. If A is weakly σ-$\mathcal{H}$-closed set, then X-A is weakly σ-$\mathcal{H}$-open and hence $X\text{-}A \subset c^*_\mu(i_\mu(c_\mu(X\text{-}A))) = X\text{-}i^*_\mu(c_\mu(i_\mu(A)))$. Therefore $i^*_\mu(c_\mu(i_\mu(A))) \subset A$. Coversely, let $i^*_\mu(c_\mu(i_\mu(A))) \subset A$. Then $X\text{-}A \subset c^*_\mu(i_\mu(c_\mu(X - A)))$. Hence X-A is weakly σ-$\mathcal{H}$-open. Thus A is weakly σ-$\mathcal{H}$-closed set. □

Definition 2.22. *A subset A of a hereditary generalized topological space $(X, \mu, \mathcal{H})$ is said to be a strong S-$\mathcal{H}$-set if $c^*_\mu(i_\mu(c_\mu(A))) = i_\mu(A)$.A subset A of a hereditary generalized topological space $(X, \mu, \mathcal{H})$ is said to be a strong $S_\mathcal{H}$-set if $A = U \cap V$, where $U \in \mu$ and V is strong S-$\mathcal{H}$-set.*

Proposition 2.23. *In a hereditary generalized topological space $(X, \mu, \mathcal{H})$, the following hold.*
(a) Every strong S-$\mathcal{H}$-set is S-$\mathcal{H}$-set,
(b) Every strong $S_\mathcal{H}$-set is $S_\mathcal{H}$-set.

Proof. (a) If A is strong S-$\mathcal{H}$-set, then $c^*_\mu(i_\mu(c_\mu(A))) = i_\mu(A)$. Now $i_\mu(A) \subset ((i_\mu(A))^* \cup i_\mu(A) = c^*_\mu(i_\mu(A))$ and $c^*_\mu(i_\mu(A)) = c^*_\mu(c^*_\mu(i_\mu(c_\mu(A)))) \subset c^*_\mu(i_\mu(c_\mu(A))) = i_\mu(A)$. Hence A is S-$\mathcal{H}$-set.
(b) Obvious. □

Proposition 2.24. *Let $(X, \mu, \mathcal{H})$ be a hereditary generalized topological space and $A \subset X$. Then A is S-$\mathcal{H}$-set if and only if $(i_\mu(A))^* \subset i_\mu(A)$.*

Proof. If A is S-$\mathcal{H}$-set, then $c^*_\mu(i_\mu(A)) = i_\mu(A)$, so $(i_\mu(A))^* \cup i_\mu(A) = i_\mu(A)$. Hence $(i_\mu(A))^* \subset i_\mu(A)$. Converse is obvious. □

Proposition 2.25. *Let $(X, \mu, \mathcal{H})$ be a hereditary generalized topological space where $\mathcal{H}$ is μ-codense and $A \subset X$. If A is S-$\mathcal{H}$-set, then $(i_\mu(A))^* = i_\mu(A)$.*

Proof. Since A is S-$\mathcal{H}$-set, $c^*_\mu(i_\mu(A)) = i_\mu(A)$. Now $c^*_\mu(i_\mu(A)) = c_\mu(i_\mu(A)) = (i_\mu(A))^*$. Therefore $i_\mu(A) = (i_\mu(A))^*$. □

Remark 2.26. *The following example shows that the converse of above theorem need not be true.*

Example 2.27. *Let $X = \{a, b, c, d, e\}$, $\mu = \{\emptyset, \{a\}, \{c\}, \{a, c\}, \{a, b, c\}, \{c, d, e\}$, $\{a, c, d, e\}, X\}$ and $\mathcal{H} = \{\emptyset, \{a\}, \{b\}, \{c\}\}$. The hereditary class $\mathcal{H}$ is not codense. If $A = \{a\}$, then A is S-$\mathcal{H}$-set and $(i_\mu(A))^* = (\{a\}^*) = \emptyset \neq i_\mu(A)$.*

Proposition 2.28. *Let $(X, \mu, \mathcal{H})$ be a hereditary generalized topological space and $A \subset X$. A μ-closed set A is strong S-$\mathcal{H}$-set if and only if A is S-$\mathcal{H}$-set.*

Proof. By propostion 2.23, every strong S-$\mathcal{H}$-set is S-$\mathcal{H}$-set. Conversely, if A is S-$\mathcal{H}$-set, then $i_\mu(A) = c^*_\mu(i_\mu(A)) = c^*_\mu(i_\mu(c_\mu(A)))$. Hence A is strong S-$\mathcal{H}$-set. □

Remark 2.29. *The following example shows that the notions weakly σ-$\mathcal{H}$-open and strong $S_\mathcal{H}$- set are independent.*

Example 2.30. *Consider a hereditary generalized topological space $(X, \mu, \mathcal{H})$ where $X = \{a, b, c, d\}$, $\mu = \{\emptyset, \{b\}, \{b, c\}, \{c, d\}, \{a, d\}, \{b, c, d\}\}$ and $\mathcal{H} = \{\emptyset, \{a\}, \{b\}\}$. Then*
(i) If $A = \{a, c\}$, then $c^*_\mu(i_\mu(c_\mu(A))) = c^*_\mu(i_\mu(\{a, c, d\})) = c^*_\mu(\{c, d\}) = \{a, c, d\} \supset A$. Hence A is weakly σ-$\mathcal{H}$-open. But A is not strong $S_\mathcal{H}$-set.
(ii) If $A = \{d\}$, then $A = U \cap V$ where $U = \{c, d\}$ is μ-open and $V = \{a, d\}$ is strong $S_\mathcal{H}$-set. But $c^*_\mu(i_\mu(c_\mu(A))) = c^*_\mu(i_\mu(\{a, d\})) = c_\mu(\emptyset) = \emptyset \not\supseteq A$, so, A is not a weakly σ-$\mathcal{H}$-open.

Theorem 2.31. *In a strong hereditary generalized topological space $(X, \mu, \mathcal{H})$, the following are equivalent:*

(a) A is μ-open.

(b) A is weakly σ-$\mathcal{H}$-open and strong $S_{\mathcal{H}}$-set.

(c) A is σ-$\mathcal{H}$-open and $S_{\mathcal{H}}$-set.

Proof. $(a) \Rightarrow (b)$ Obvious.

$(b) \Rightarrow (a)$ If A is weakly σ-$\mathcal{H}$-open and also a strong $S_{\mathcal{H}}$-set. Then $A \subset c^*_\mu(i_\mu(c_\mu(A))) = c^*_\mu(i_\mu(c_\mu(U \cap V)))$, where $U \in \mu$ and V is strong S-$\mathcal{H}$-set. Hence $A \subset U \cap A \subset U \cap (c^*_\mu(i_\mu(c_\mu(U))) \cap c^*_\mu(i_\mu(c_\mu(V)))) = U \cap i_\mu(V) = i_\mu(U \cap V) = i_\mu(A)$. This shows that A is μ-open.

$(b) \Rightarrow (c)$ Obvious. □

3. Generalized Weakly $(\sigma_{\mathcal{H}}, \lambda)$-continuous Functions

Definition 3.1. *A function $f : (X, \mu, \mathcal{H}) \to (Y, \lambda)$ is said to be weakly $(\sigma_{\mathcal{H}}, \lambda)$-continuous (resp. strong $(S_{\mathcal{H}}, \lambda)$-continuous) if $f^{-1}(V)$ is weakly σ-$\mathcal{H}$-open (resp. strong $S_{\mathcal{H}}$-set) in $(X, \mu, \mathcal{H})$ for every μ-open V of (Y, λ).*

Proposition 3.2. *For a function $f : (X, \mu, \mathcal{H}) \to (Y, \lambda)$, the following hold.*

(a) Every (μ, λ)-continuous is weakly $(\sigma_{\mathcal{H}}, \lambda)$-continuous.

(b) Every $(\alpha_{\mathcal{H}}, \lambda)$-continuous is weakly $(\sigma_{\mathcal{H}}, \lambda)$-continuous.

(c) Every strong $(\beta_{\mathcal{H}}, \lambda)$-continuous is weakly $(\sigma_{\mathcal{H}}, \lambda)$-continuous.

(d) Every $(\sigma_{\mathcal{H}}, \lambda)$-continuous is weakly $(\sigma_{\mathcal{H}}, \lambda)$-continuous.

(e) Every $(\pi_{\mathcal{H}}, \lambda)$-continuous is weakly $(\sigma_{\mathcal{H}}, \lambda)$-continuous.

Proof. It follows from Proposition 2.2. □

Theorem 3.3. *For a function $f : (X, \mu, \mathcal{H}) \to (Y, \lambda)$, the following equivalent.*

(a) f is weakly $(\sigma_{\mathcal{H}}, \lambda)$-continuous.

(b) For each $x \in X$ and each λ-open V containing $f(x)$, there exists weakly σ-$\mathcal{H}$-open U such that $f(U) \subset V$.

Proof. Let $x \in X$ and V be λ-open set of Y containing $f(x)$. Take $W = f^{-1}(V)$, then by Definition W is weakly σ-$\mathcal{H}$-open containing x and $f(W) \subset V$.

Conversely, let F be a λ-closed set of Y. Take $V = Y - F$, then V is μ-open in Y. Let $x \in f^{-1}(V)$, by hypothesis , there exists a weakly σ-$\mathcal{H}$-open W of X containing x such that $f(W) \subset V$. Thus, we obtain $x \in W \subset c^*_\mu(i_\mu(c_\mu(W))) \subset c^*_\mu(i_\mu(c_\mu(f^{-1}(V))))$ and hence $f^{-1}(V) \subset c^*_\mu(i_\mu(c_\mu(V)))$. This shows that $f^{-1}(V)$ is weakly σ-$\mathcal{H}$-open in X. Hence $f^{-1}(F) = X - f^{-1}(Y - F) = X - f^{-1}(V)$ is weakly σ-$\mathcal{H}$-closed set in X. □

Theorem 3.4. *Let $f : (X, \mu, \mathcal{H}_1) \to (Y, \lambda, \mathcal{H}_2)$ and $g : (Y, \lambda, \mathcal{H}_2) \to (Z, \eta)$ be two functions, where $\mathcal{H}_1$ and $\mathcal{H}_2$ are hereditary classes on X, Y and Z respectively. Then $g \circ f$ is weakly $(\sigma_{\mathcal{H}}, \lambda)$-continuous if f is weakly $(\sigma_{\mathcal{H}}, \lambda)$-continuous and g is (λ, η)-continuous.*

Proof. Let U be any η-open in Z. Since g is (λ, η)-continuous, $g^{-1}(U)$ is λ-closed in Y. Since f is weakly $(\sigma_{\mathcal{H}}, \lambda)$-continuous, $f^{-1}(g^{-1}(U)) = (g \circ f)^{-1}(U)$ is weakly σ-$\mathcal{H}$-open in X. Hence $g \circ f$ is weakly $(\sigma_{\mathcal{H}}, \lambda)$-continuous. □

Proposition 3.5. *Let $(X, \mu, \mathcal{H})$ and (Y, λ) be hereditary generalized topological space and generalized topology respectively. Then a function $f : (X, \mu, \mathcal{H}) \to (Y, \lambda)$, the following are equivalent.*

(a) f is (μ, λ)-continuous.

(b) f *is* $(\sigma_{\mathcal{H}}, \lambda)$*-continuous and* $(S_{\mathcal{H}}, \lambda)$*-continuous.*

(c) f *is weakly* $(\sigma_{\mathcal{H}}, \lambda)$*-continuous and strong* $(S_{\mathcal{H}}, \lambda)$*-continuous.*

Proof. It is obvious from Theorem 2.31. □

Theorem 3.6. *A function* $f : (X, \mu, \mathcal{H}) \to (Y, \lambda)$ *is weakly* $(\sigma_{\mathcal{H}}, \lambda)$*-continuous if and only if* $f^{-1}(U)$ *is weakly* σ*-*$\mathcal{H}$*-closed in* $(X, \mu, \mathcal{H})$*, for every* λ*-closed* U *in* (Y, λ)*.*

Proof. Let f be weakly $(\sigma_{\mathcal{H}}, \lambda)$-continuous and F be a λ-closed set in (Y, λ). Then $Y - F$ is λ-open in (Y, λ). Since f is weakly $(\sigma_{\mathcal{H}}, \lambda)$-continuous, $f^{-1}(Y - F)$ is weakly σ-$\mathcal{H}$-open in $(X, \mu, \mathcal{H})$. But $f^{-1}(Y - F) = X - f^{-1}(F)$ and so $f^{-1}(F)$ is weakly σ-$\mathcal{H}$-closed in $(X, \mu, \mathcal{H})$. Coversely, assume that $f^{-1}(F)$ is weakly σ-$\mathcal{H}$-closed in $(X, \mu, \mathcal{H})$ for every λ-closed set in F in (Y, λ). Let V be a λ-open in (Y, λ). Then $Y - V$ is λ-closed in (Y, λ) and by hypothesis $f^{-1}(Y - F)$ is weakly σ-$\mathcal{H}$-closed in $(X, \mu, \mathcal{H})$. Since $f^{-1}(Y - V) = X - f^{-1}(V)$, we have $f^{-1}(V)$ is weakly σ-$\mathcal{H}$-open in $(X, \mu, \mathcal{H})$, and so f is weakly $(\sigma_{\mathcal{H}}, \lambda)$-continuous. □

Definition 3.7. *Let* x *be any element of a hereditary generalized topological space* $(X, \mu, \mathcal{H})$ *and* $V \subset X$*. Then* V *is said to be weakly* $\sigma_{\mathcal{H}}$*-neighbourhood of* x *in* X *if there exists weakly* σ*-*$\mathcal{H}$*-open* U *of* X *such that* $x \in U \subset V$*.*

Theorem 3.8. *For a function* $f : (X, \mu, \mathcal{H}) \to (Y, \lambda)$*, the following are equivalent.*

(a) f *is weakly* $(\sigma_{\mathcal{H}}, \lambda)$*-continuous.*

(b) *the inverse image of each* λ*-closed set is weakly* σ*-*$\mathcal{H}$*-closed.*

(c) *for each* x *in* X*, the inverse of every* λ*-neighbourhood of* $f(x)$ *is weakly* $\sigma_{\mathcal{H}}$*-neighbourhood of* x *in* X*.*

(d) *For each* x *in* X *and every* λ*-open* U *containing* $f(x)$*, there exists weakly* σ*-*$\mathcal{H}$*-open* V *containing* x *such that* $f(V) \subset U$*.*

(e) $f(c_\mu(A)) \subset c_\lambda(f(A))$ *for every subset* A *of* X*.*

(f) $c_\mu(f^{-1}(B)) \subset f^{-1}(c_\lambda(B))$ *for every subset* B *of* Y*.*

Proof. $(a) \Leftrightarrow (b)$ This follows from Theorem 3.6.

$(b) \Rightarrow (c)$. Let $x \in X$. Assume that V be a neighbourhood of $f(x)$. Then there exists a λ-open U in Y such that $f(x) \in U \subset V$. Consequently $f^{-1}(U)$ is weakly σ-$\mathcal{H}$-open in X and $x \in f^{-1}(U) \subset f^{-1}(V)$ is weakly $\sigma_{\mathcal{H}}$-neighbourhood of x.

$(c) \Rightarrow (d)$. Let $x \in X$ and U be a neighbourhood of $f(x)$. Then by hypothesis, $V = f^{-1}(U)$ is weakly $\sigma_{\mathcal{H}}$-neighbourhood of x and $f(V) = f(f^{-1}(U)) \subset U$.

$(d) \Rightarrow (e)$. Let A be a subset of X such that $f(x) \notin c_\lambda(f(A))$. Then, there exists a λ-open subset V of Y containing $f(x)$ such that $V \cap f(A) = \emptyset$. By hypothesis there exists a μ-open U such that $f(x) \in f(U) \subset V$. Hence $f(U) \cap f(A) = \emptyset$ implies $U \cap A = \emptyset$. Consequently $x \notin c_\mu(A)$ and $f(x) \notin f(c_\mu(A))$. Hence $f(c_\mu(A)) \subset c_\lambda(f(A))$.

$(e) \Rightarrow (f)$. Let B be a subset of Y. By hypothesis, we obtain $f(c_\mu(f^{-1}(B))) \subset c_\lambda(f(f^{-1}(B)))$. Thus $c_\mu(f^{-1}(B)) \subset f^{-1}(c_\lambda(B))$.

$(e) \Rightarrow (a)$. Let F be a λ-closed subset of Y. Since $c_\lambda(F) = F$ and by hypothesis $f(c_\mu(f^{-1}(F))) \subset c_\lambda(f(f^{-1}(F))) \subset c_\lambda(F) = F$. This shows that $c_\mu(f^{-1}(F)) \subset f^{-1}(F)$ and so $f^{-1}(F)$ is weakly σ-$\mathcal{H}$-closed. □

Definition 3.9. *A function* $f : (X, \mu) \to (Y, \lambda, \mathcal{H})$ *is said to be weakly* $(\sigma, \lambda_{\mathcal{H}})$*-open (resp. weakly* $(\sigma, \lambda_{\mathcal{H}})$*-closed) if the image of every* μ*-open (resp.* μ*-closed) in* (X, μ) *is weakly* σ*-*$\mathcal{H}$*-open (resp. weakly* σ*-*$\mathcal{H}$*-closed) in* $(Y, \lambda, \mathcal{H})$*.*

Proposition 3.10. *For a bijective function* $f : (X, \mu) \to (Y, \lambda, \mathcal{H})$*, the following hold.*

(a) f^{-1} *is weakly* $(\sigma_{\mathcal{H}}, \lambda)$*-continuous.*

(b) f *is weakly* $(\sigma, \lambda_{\mathcal{H}})$*-open.*

(c) f *is weakly* $(\sigma, \lambda_{\mathcal{H}})$*-closed.*

Theorem 3.11. *A function $f : (X, \mu) \to (Y, \lambda, \mathcal{H})$ is weakly $(\sigma, \lambda_{\mathcal{H}})$-open if and only if for each subset $W \subset Y$ and each μ-closed F of X containing $f^{-1}(W)$, there exists weakly σ-$\mathcal{H}$-closed $H \subset Y$ containing W such that $f^{-1}(H) \subset F$.*

Proof. Let $H = Y - f(X - F)$. Since $f^{-1}(W) \subset F$, we have $f(X - F) \subset Y - W$. Since f is weakly $(\sigma, \lambda_{\mathcal{H}})$-open, then H is weakly σ-$\mathcal{H}$-closed and $f^{-1}(H) = X - f^{-1}(f(X - F)) \subset X - (X - F) = F$.
Conversely, let U be any μ-open of X and $W = Y - f(U)$. Then $f^{-1}(W) = X - f^{-1}(f(U)) \subset X - U$ and $X - U$ is μ-closed. By the hypothesis, there exists weakly σ-$\mathcal{H}$-closed H of Y containing W such that $f^{-1}(H) \subset X - U$. Then, we have $f^{-1}(H) \cap U = \emptyset$ and $H \cap f(U) = \emptyset$. Thus, we obtain $Y - f(U) \supset H \supset W = Y - f(U)$ and $f(U)$ is weakly σ-$\mathcal{H}$-open in Y. Hence f is weakly $(\sigma, \lambda_{\mathcal{H}})$-open. □

Corollary 3.12. *If $f : (X, \mu) \to (Y, \lambda, \mathcal{H})$ is weakly $(\sigma, \lambda_{\mathcal{H}})$-open, then $f^{-1}(i^*_\lambda(c_\lambda(i_\lambda(B)))) \subset c_\mu(f^{-1}(B))$ for each subset $B \subset Y$,*

Proof. Let B be a subset of Y, then $c_\mu(f^{-1}(B))$ is μ-closed in X. By Theorem 3.11, there exists weakly σ-$\mathcal{H}$-closed $H \subset Y$ containing B such that $f^{-1}(H) \subset c_\mu(f^{-1}(B))$. Since $Y - H$ is weakly σ-$\mathcal{H}$-open, $f^{-1}(Y - H) \subset f^{-1}(c^*_\lambda(i_\lambda(c_\lambda(Y - H))))$ and $X - f^{-1}(H) \subset f^{-1}(Y - i^*_\lambda(c_\lambda(i_\lambda(H)))) = X - f^{-1}(i^*_\lambda(c_\lambda(i_\lambda(H))))$. Hence we obtain that $f^{-1}(i^*_\lambda(c_\lambda(i_\lambda(B)))) \subset f^{-1}(i^*_\lambda(c_\lambda(i_\lambda(H)))) \subset f^{-1}(H) \subset c_\mu(f^{-1}(B))$. Therefore, we have $f^{-1}(i^*_\lambda(c_\lambda(i_\lambda(B)))) \subset c_\mu(f^{-1}(B))$. □

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