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Best Proximity Point Theorems for α-Rational Cyclic Contraction

Seminar Paper*

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Abstract: By using the classical fundamental cyclic contraction and, we introduce the new notion of α-rational cyclic contraction, and then we establish some best proximity point theorems for non-self mapping in the frame work of complete metric space. Our results generalized and improve some main results in the literature. Some examples are given to support our main results.

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1. Introduction

A fundamental result in fixed point theory is the Banach Contraction Principle. One kind of a generalization of the Banach Contraction Principle is the notation of cyclical maps [7]. Fixed point theory is an important tool for solving equations $Tx = x$ for mappings T defined on subsets of metric spaces or normed spaces. Because a non-self mapping $T : A \to B$ does not necessarily have a fixed point, one often attempts to find an element x which is in some sense closest to Tx. Best proximity point theorems are relevant in this perspective. The notation of best proximity point is introduced in [5]. This definition is more general than the notation of cyclical maps, in sense that if the sets intersect, then every best proximity point is a fixed point. A sufficient condition for the uniqueness of the best proximity points in uniformly convex Banach spaces is given in [5].

On the other hand, given non-empty subsets A and B of X and a non-self map- ping T from A to B, one can perceive that the equation $Tx = x$ is improbable to have a solution, for a solution of the preceding equation requires the equality between an element in A and an element in B. Consequently, one speculates to determine an approximate solution x that is optimal in the sense that the distance between x and $T x$ is minimum. In fact, in the case that $Tx = x$ has no solution, one attempts to determine an element x that is in close proximity to $T x$. On account of the facts that minimality of the distance between x and Tx ascertains the highest adjacency between x and Tx and that distance between x and Tx is at least the distance between A and B for all $x \in A$, an ideal scenario is to discover an element x for which the distance

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between x and Tx assumes the least possible value that is the distance between A and B. Such an optimal approximate solution x, for which the distance between x and $T x$ equals the distance between A and B, is denominated as a best proximity point of the mapping T . It is remarked that a best proximity point serves as an optimal approximate solution to the equation $Tx = x$. Best proximity point theorems are those results that present sufficient conditions for the existence of a best proximity point and algorithms for finding best proximity points. It is interesting to see that best proximity point theorems generalize fixed point theorems in a natural way. In fact, when the mapping under consideration is a self-mapping, a best proximity point boils down to a fixed point. Some interesting best proximity point theorems in the setting of metric spaces or normed linear spaces can be found in [2, 3, 5, 6, 9, 10, 14, 13, 15, 16]. It is interesting that in all investigated conditions for the existence of best proximity the distances between sets are equal. We first define the cyclic map and the best proximity point.

Definition 1.1. *Let A and B be non-empty subsets of a metric space $(X,\ d)$ and $T : A \cup B \to A \cup B$. Then T is called cyclic map if s3$T(A) \subset B$ and $T(B) \subset A$. A point $x \in A \cup B$ is called a best proximity point if $d(x, Tx) = d(A, B)$, where $d(A, B) = \inf\{d(x, y) : x \in A, y \in B\}$.*

In 2003, Kirk et al. [7] gave the following fixed point theorem for a cyclic map.

Theorem 1.2. *Let A and B be nonempty closed subsets of a complete metric space (X, d). Suppose that $T : A\cup B \to A\cup B$ is a cyclic map such that*

$$d(Tx, Ty) \leq kd(x, y)$$

for all $x \in A,\ y \in B$. If $k \in [0, 1)$, then T has a unique fixed point in $A \cup B$.

In 2005, Eldred et al. [4] proved the existence of a best proximity point for relatively non-expansive mappings using the notion of proximal normal structure. In 2006, Eldred and Veeramani [5] proved the following existence theorem.

Theorem 1.3. *Let A and B be nonempty closed convex subsets of a uniformly convex Banach space. Suppose $T : A \cup B \to A \cup B$ is a cyclic contraction, that is, $T(A) \subset B$ and $T(B) \subset A$ and there exists $k \in (0, 1)$ such that*

$$d(Tx, Ty) \leq kd(x, y) + (1 - k)d(A, B), \tag{1}$$

for every $x \in A,\ y \in B$. Then there exists a unique best proximity point in A. Further, for each $x \in A$, $\{f^{2n}x\}$ converges to the best proximity point.

Later, best proximity point theorems for various types of contractions have been obtained in [10, 12, 1, 1, 6]. Particularly, in [12], the authors prove some best proximity point theorems for K-cyclic mappings and C-cyclic mappings in the frameworks of metric spaces and uniformly convex Banach spaces, thereby furnishing an optimal approximate solution to the equations of the form $Tx = x$, where T is a non-self-K-cyclic mapping or a non-self-C-cyclic mapping.

In this paper, we have found a new type of contraction, which warrants the convergence and existence of the best proximity points for sets with different distances between them. This new type of a map we have called a α-rational cyclic contraction map.

2. Preliminaries

In this section, we first define the cyclic contraction and new notion of rational cyclic contraction.

Let $(X,\ d)$ be a complete metric space and $A,\ B$ be nonempty subsets of X. A mapping $T : X \to X$ is a contraction if and only if for each $x,\ y \in X$ there exists a constant $k \in (0,\ 1)$ such that $d(T\ x,\ T\ y) \leq kd(x,\ y)$. Let $T : A \cup B \to A \cup B$ such that $T\ (A) \subset B$ and $T\ (B) \subset A$ we say that

(i) T is cyclic contraction if

$$d(Tx, Ty) \leq \alpha d(x, y) + (1-\alpha)d(A, B),$$

for some $\alpha \in (0, 1)$ and for all $x \in A$ and $y \in B$ where $d(A, B) = \inf\{d(x, y) : x \in A, y \in B\}$.

Definition 2.1 ([8]). *Let (X, d) be a complete metric space and let A and B be nonempty subsets of X. Then a cyclic operator $T : A \cup B \to A \cup B$ is called weak cyclic Kannan contraction if it satisfies the following condition*

$$d(Tx, Ty) \leq k[d(x, Tx) + d(y, Ty)] + (1-2k)d(A, B), \tag{2}$$

where $k \in [0,\ 1/2)$, for all $x \in A$ and $y \in B$.

Definition 2.2. *A pair of mappings $T : A \cup B \to A \cup B$ is said to form a $\alpha-$rational cyclic contraction mapping between A and B if there exists a nonnegative real number $0 \leq \alpha < 1$ such that*

$$d(Tx, Ty) \leq \alpha \frac{[d(x, Tx)]^2}{d(x, Tx) + d(y, Tx)} + (1-\alpha)d(A, B) \tag{3}$$

for all $x \in A$ and $y \in B$.

Definition 2.3. *A subset K of a metric space is said to be boundedly compact if every bounded sequence in K has a subsequence converging to some element in K.*

3. α-rational Cyclic Contraction

In this section, first, we establish the following convergence theorem for $\alpha-$rational cyclic contraction, which is one of the main results in this paper.

Theorem 3.1. *Let A and B be two nonempty, closed subsets of a complete metric space (X, d) and the mappings $T : A \cup B \to T : A \cup B$ satisfy $\alpha-$rational cyclic contraction, then there exists a sequence $\{x_n\}$ such that*

$$\lim_{n \to \infty} d(x_n, x_{n+1}) = d(A, B).$$

Proof. Since T is satisfy a $\alpha-$rational cyclic contraction such that

$$d(Tx, Ty) \leq \alpha \frac{[d(x, Tx)]^2}{d(x, Tx) + d(y, Tx)} + (1-\alpha)d(A, B) \tag{4}$$

for all x ∈ A and y ∈ B. Suppose $x_0 \in A \cup B$ be given. Define a sequence $\{x_n\}$ by $x_{n+1} = Tx_n$ for all $n \in N$. From (4), we obtain

$$\begin{aligned} d(x_{n+1}, x_{n+2}) &= d(Tx_n, Tx_{n+1}) \\ &\leq \alpha \frac{[d(x_n, Tx_n)]^2}{d(x_n, Tx_n) + d(x_{n+1}, Tx_n)} + (1-\alpha)d(A, B) \\ &\leq \alpha \frac{[d(x_n, x_{n+1})]^2}{d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+1})} + (1-\alpha)d(A, B) \\ &\leq \alpha \frac{[d(x_n, x_{n+1})]^2}{d(x_n, x_{n+1}) + 0} + (1-\alpha)d(A, B), \end{aligned}$$

which implies that,

$$d(x_{n+1}, x_{n+2}) \leq \alpha d(x_n, x_{n+1}) + (1-\alpha)d(A, B), \tag{5}$$

Similarly, we get

$$\begin{aligned} d(x_{n+2}, x_{n+3}) &= d(Tx_{n+1}, Tx_{n+2}) \\ &\leq \alpha \frac{[d(x_{n+1}, Tx_{n+1})]^2}{d(x_{n+1}, Tx_{n+1}) + d(x_{n+2}, Tx_{n+1})} + (1-\alpha)d(A, B) \\ &\leq \alpha \frac{[d(x_{n+1}, x_{n+2})]^2}{d(x_{n+1}, x_{n+2}) + d(x_{n+2}, x_{n+2})} + (1-\alpha)d(A, B) \\ &\leq \alpha \frac{[d(x_{n+1}, x_{n+2})]^2}{d(x_{n+1}, x_{n+2}) + 0} + (1-\alpha)d(A, B) \\ &\leq \alpha d(x_{n+1}, x_{n+2}) + (1-\alpha)d(A, B) \end{aligned} \tag{6}$$

By this way if we continue, we concluded that from (5) and (6), we have

$$d(x_{n+2}, x_{n+3}) \leq \alpha^2 d(x_n, x_{n+1}) + (1-\alpha^2)d(A, B).$$

Hence inductively, we have

$$\begin{aligned} d(x_n, x_{n+1}) &\leq \alpha d(x_n, x_{n-1}) + (1-\alpha)d(A, B) \\ &\leq \alpha^2 d(x_{n-1}, x_{n-2}) + (1-\alpha^2)d(A, B) \\ &\leq \dots \\ &\leq \alpha^n d(x_0, x_1) + (1-\alpha^n)d(A, B) \end{aligned}$$

Since $\alpha \in [0, 1)$, we have $\lim_{n\to\infty} \alpha^n = 0$, So the last inequality implies that $\lim_{n\to\infty} d(x_n, x_{n+1}) = d(A, B)$. This completes the proof. □

Theorem 3.2. *Let A and B be two nonempty, closed subsets of a complete metric space (X, d) and the mapping $T : A \cup B \to T : A \cup B$ satisfy $\alpha-$rational cyclic contraction between A and B, let $x_0 \in A$ and defined an iteration $x_{n+1} = Tx_n$. Let the sequence $\{x^{2n}\}$ has a subsequence converging to some element in A. Then there exists $x \in A$ such that $d(x, T x) = d(A, B)$.*

Proof. Suppose that a sequence $\{x_{2n_k}\}$ be a subsequence of $\{x_{2n}\}$ converges to some element x in A. Since $d(x_{2n_k}, x_{2n_k-1})$ converges to $d(A, B)$ (From Theorem 3.1). Furthermore,

$$\begin{aligned} d(A, B) &\leq d(x, x_{2n_k-1}) \\ &\leq d(x, x_{2n_k}) + d(x_{2n_k}, x_{2n_k-1}) \\ &\leq d(x, x_{2n_k}) + d(A, B). \end{aligned}$$

Therefore $d(x, x_{2n_k-1}) \to d(A, B)$. Since T is a α-rational cyclic contraction, it follows that

$$\begin{aligned} d(A, B) &\leq d(x_{2n_k}, Tx) \\ &= d(Tx_{2n_k-1}, Tx) \\ &\leq \alpha \frac{[d(x_{2n_k-1}, Tx_{2n_k-1})]^2}{d(x_{2n_k-1}, Tx_{2n_k-1}) + d(x, Tx_{2n_k-1})} + (1-\alpha)d(A, B) \\ &\leq \alpha \frac{[d(x_{2n_k-1}, x_{2n_k})]^2}{d(x_{2n_k-1}, x_{2n_k}) + d(x, x_{2n_k})} + (1-\alpha)d(A, B) \\ &\leq \alpha d(x_{2n_k-1}, x_{2n_k}) + (1-\alpha)d(A, B). \end{aligned}$$

Taking $n \to \infty$ above inequality and from Theorem 3.1 that $d(x_{2n_k}, x_{2n_k-1}) \to d(A,B)$. We have $d(x,Tx) = d(A,B)$, that is x is a best proximity point of T. This completes the proof. □

Theorem 3.3. *Let A and B be two nonempty, closed subsets of a complete metric space (X, d). Suppose that the mappings $T : A \cup B \to T : A \cup B$ form a rational cyclic contraction between A and B, for each fixed $x_0 \in A$, let defined iteration $x_{n+1} = Tx_n$. Let the sequence $\{x_{2n}\}$ has a subsequence converging to some element in A. Then the sequence $\{x_n\}$ is bounded.*

Proof. Assume that a sequence $\{x_{2n_k}\}$ be a subsequence of $\{x_{2n}\}$ converges to some element x in A and it follows from Theorem 3.1, that $\{d(x_{2n-1}, x_{2n})\}$ is convergent and hence it is bounded. Since $T : A \cup B \to A \cup B$ satisfy a α-rational cyclic contraction, we have

$$\begin{aligned} d(x_{2n_k}, Tx_0) &= d(Tx_{2n_k-1}, Tx_0) \\ &\leq \alpha \frac{[d(x_{2n_k-1}, Tx_{2n_k-1})]^2}{d(x_{2n_k-1}, Tx_{2n_k-1}) + d(x_0, Tx_{2n_k-1})} + (1-\alpha)d(A,B) \\ &\leq \alpha \frac{[d(x_{2n_k-1}, x_{2n_k})]^2}{d(x_{2n_k-1}, x_{2n_k}) + d(x_0, x_{2n_k})} + (1-\alpha)d(A,B) \\ &\leq \alpha d(x_{2n_k-1}, x_{2n_k}) + (1-\alpha)d(A,B) \\ &\leq [\alpha d(x_{2n_k-1}, x_{2n_k}) - d(A,B)] + d(A,B). \end{aligned}$$

Since $\alpha \in [0,1)$, therefore, the sequence $\{x_{2n}\}$ is bounded. Hence the sequence $\{x_n\}$ is also bounded. This completes the proof. □

Following examples illustrating our main results.

Example 3.4. *Consider the usual metric space $d(x,y) = jx - yj$, for all $x, y \in X$. Let $X = R$. Suppose $A = [1,2]$ and $B = [-2,-1]$, then $d(A,B) = 2$. Define a mapping $T : A \cup B \to A \cup B$ as follows:*

$$T(x) = \begin{cases} \frac{-1-x}{2}, & \text{if } x \in A \\ \frac{-1+x}{2}, & \text{if } x \in B. \end{cases}$$

It is clear that $T(A) \subset B$ and the inequality (3) holds. Hence 2 is a best proximity point of T. So that T is an α-rational cyclic contraction.

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