International Journal of Current Research in Science and Technology
Volume 2, Issue 1 (2016), 55–60. (Special Issue)
ISSN: 2394-5745
Available Online: http://ijcrst.in/

International Journal of Current Research in Science and Technology

Iterative Algorithm for Solution of m-accretive Operator in Uniformly Smooth Banach Spaces

Seminar Paper*

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Abstract: The purpose of this paper is to study a composite iterative scheme for approximating solution of m-accretive operator in a uniformly convex and uniformly smooth Banach space using the resolvent and retraction technique. The result presented in this paper thus improve and extend the corresponding results of Kim and Xu [9], Qin and Su [12] and the references therein, to a better iterative scheme and that of Marino and Xu [10], Takahashi [16] and the references therein, to a more general Banach space.

MSC: 47H06, 47H09.

Keywords: m-accretive operator, uniformly convex Banach space, uniformly smooth, composite iteration, resolvent, retraction.

1. Introduction

Let E be a real Banach space with dual E^*. The *normalized duality mapping* from E to 2^{E^*} is defined by

$$J(x) := \{x^* \in E^* : \langle x, x^* \rangle = \|x\|^2, \|x\| = \|x^*\|\},$$

where $\langle ., . \rangle$ denotes the duality pairing between the elements of E and E^*.

Definition 1.1 ([3]). *A mapping $A\colon D(A) \subseteq E \to E$ is said to be accretive if for all $x, y \in E$, there exists $j(x-y) \in J(x-y)$ such that $\langle Ax - Ay, j(x-y)\rangle \geq 0$. If E is a Hilbert space, accretive operators are also called monotone. An operator A is called m-accretive if it is accretive and $\mathcal{R}(I + rA)$, the range of $(I + rA)$, is E for all $r > 0$; and A is said to satisfy the range condition if $\overline{D(A)} \subseteq \mathcal{R}(I + rA), \forall\, r > 0$, where $\mathcal{R}(I + rA) = \{z + rAz : z \in E, Az \neq \phi\}$.*

Closely related to the class of accretive mappings is the class of pseudocontractive mappings.

Definition 1.2 ([4]). *The mapping $T : E \to E$ is called pseudocontractive if for all $x, y \in E$, there exists $j(x-y) \in J(x-y)$ such that $\langle Tx - Ty, j(x-y)\rangle \leq \|x-y\|^2$.*

* *Proceedings : UGC Sponsored National Seminar on* `Value and Importance of Mathematical Physics` *held on 05.12.2015, organized by Department of Mathematics and Physics, Government Rajeev Lochan College, Rajim, Gariaband (Chhattisgarh), India.*

The mapping T is pseudocontractive if and only if $(I - T)$ is accretive. It is well known that if A is accretive [6], then $J_r := (I + rA)^{-1}$ is a nonexpansive single-valued mapping from $\mathcal{R}(I + rA)$ to $D(A)$ and $\mathcal{F}(J_r) = \mathcal{N}(A)$, for each $r > 0$, where $\mathcal{N}(\mathcal{A}) := \{x \in D(A) : Ax = 0\} = A^{-1}(0)$ and $\mathcal{F}(J_r) := \{x \in D(A) : J_r x = x\}$. Here we also note that x^* is a zero of the accretive mapping A if and only if it is a fixed point of the pseudocontractive mapping $T := I - A$.

Also if A is accretive then the solutions of the equation $Ax = 0$ correspond to the equilibrium points of some evolution systems [18]. Consequently, considerable research efforts, especially within the past 15 years or so, have been devoted to iterative methods for approximating the zeros of A, when A is accretive. Let K be a closed convex subset of a real Banach space E. A mapping $T : K \to E$ is called a contraction mapping if there exists $L \in [0, 1)$ such that $\|Tx - Ty\| \leq L\|x - y\|$, for all $x, y \in K$. If $L = 1$, then T is called nonexpansive.

Clearly the class of nonexpansive mappings is a subset of the class of pseudocontractive mappings. In 1976, Rockafellar [14] introduced a proximal point algorithm in a Hilbert space for a maximal monotone operator: For any $x_0 \in H$, the sequence $\{x_n\}$ defined by

$$x_{n+1} = J_{r_n} x_n, \forall n \in N \tag{1}$$

where $\{r_n\} \subset (0, \infty)$ satisfies $\liminf_{n\to\infty} r_n > 0$, converges weakly to an element of $A^{-1}0 = \{x \in C : 0 \in Ax\}$. The weak and strong convergence of the sequence $\{x_n\}$ have been extensively discussed for in Hilbert spaces and in Banach spaces (see e.g. [15] and the references therein). Whereas in 2000, Kamimura and Takahashi [8] modified the above results and proved a strong convergence theorem for a monotone operator in a Hilbert space as: For A a maximal monotone operator and $J_r = (I + rA)^{-1}$ for all $r > 0$, let the sequence $\{x_n\}$ be defined by

$$x_{n+1} = \alpha_n x + (1 - \alpha_n) J_{r_n} x_n, n \geq 0, \tag{2}$$

where $\{\alpha_n\} \subset [0, 1]$ and $\{r_n\} \subset (0, \infty]$ satisfy the conditions (C1) $\lim\limits_{n\to\infty} \alpha_n = 0$ and (C2) $\sum\limits_{n=0}^{\infty} \alpha_n = +\infty$ and (C3) $\lim\limits_{n\to\infty} r_n = +\infty$. Then the iterative sequence $\{x_n\}$ converges strongly to some $A^{-1}0$. This result was extended in 2005 by Kim and Xu [9] to a uniformly smooth Banach space E giving the result: Suppose that A is an m-accretive operator, and $J_r := (I + rA)^{-1}$ for all $r > 0$, and the sequence $\{x_n\}$ is defined by (2), where $\{\alpha_n\} \subset [0, 1]$ and $\{r_n\} \subset (0, \infty]$ satisfy the following conditions: (C1), (C2) and (C3) $\sum\limits_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < +\infty$; (C4) $\sum\limits_{n=1}^{\infty} \left|1 - \frac{r_{n+1}}{r_n}\right| < +\infty$. Then $\{x_n\}$ converges strongly to a zero of A. This work was further extended by Xu [9] in the framework of Reflexive Banach space having weakly continuous duality map.

All these results hold when the operator is defined on the whole of E. But generally, the domain of $A, D(A)$, is a proper subset of E. In such a situation, these iteration processes may not even be well defined. In the case that $E = H$, a Hilbert space, this problem has been overcome by introducing the proximity map, $P_K : H \to K$, where K is a closed convex subset of H and P_K is the map which sends each $x \in H$ to its nearest point in K. It is well known that in H, the map P_K is nonexpansive and this fact is central in using the proximity map. Unfortunately, the fact that P_K is nonexpansive in Hilbert spaces also characterizes Hilbert spaces so that this fact is not available in general Banach spaces.

Thus in this paper, we show the convergence of an iterative algorithm in uniformly convex and uniformly smooth Banach space when the domain of m-accretive operator, $D(A)$ is a proper subset of the space E using retraction principle. The algorithm is defined as: For any $x_0 \in D(A)$, let the sequence $\{x_n\}$ be generated by

$$\begin{cases} y_n & = \alpha_n x_0 + (1-\alpha_n) J_{r_n} x_n \\ C_n & = \{z \in E : \|y_n - z\|^2 \leq \alpha_n \|x_n - z\|^2 + 2\alpha_n \langle x_0 - z, j(y_n - z)\rangle\} \\ H_n & = \{z \in D(A) : \langle x_n - x_0, j(x_n - z)\rangle \leq 0\} \\ x_{n+1} & = Q_{C_n \bigcap H_n} x_0 \end{cases} \tag{3}$$

Thus the purpose of this paper is to prove that the sequence $\{x_n\}$ defined by the composite iteration scheme (3) converges strongly to a zero of m-accretive operator in a uniformly convex and uniformly smooth Banach space, thus generalizing and extending of the results of Kamimura and Takahashi [8], Kim and Xu [9], Qin and Su [12] and the references therein to a better iterative scheme and that of Marino and Xu [10], Takahashi [16] and the references therein, to a more general Banach space.

2. Preliminaries

Definition 2.1. *A Banach space E is called smooth [6] if, for every $x \in E$ with $\|x\| = 1$; there exists unique $j \in E^*$ such that $\|j\| = j(x) = 1$. The modulus of smoothness of E is the function $\rho_E : [0,\infty) \to [0,\infty)$ defined by $\rho_E(\tau) = \sup\{\frac{1}{2}(\|x+y\|+\|x-y\|)-1 : x, y \in E, \|x\| = 1, \|y\|\tau\}$. The Banach space E is called uniformly smooth [17] if $\lim\limits_{\tau\to 0} \frac{\rho_E(\tau)}{\tau} = 0$.*

Definition 2.2. *The Banach space E is called uniformly convex if given any $\epsilon > 0$, there exists $\delta > 0$ such that for all $x, y \in E$ with $\|x\| \leq 1$, $\|y\| \leq 1$ and $\|x-y\| \geq \epsilon$ we have $\|\frac{1}{2}(x+y)\| \leq 1-\delta$.*

It is known that every uniformly convex Banach space is reflexive. Let $K \subset E$ be closed and convex and Q be a mapping of E onto K. Then Q is said to be sunny if

$$Q(Q(x) + t(x - Q(x))) = Q(x), \text{ whenever } Q(x) + t(x - Q(x)) \in E \text{ for } x \in E \text{ and } t \geq 0.$$

A mapping Q of E into E is said to be a retraction if $Q^2 = Q$. If a mapping Q is a retraction, then $Q(z) = z$ for every $z \in R(Q)$,where $R(Q)$ is the range of Q. A subset K of E is said to be a sunny nonexpansive retract of E if there exists a sunny nonexpansive retraction of E onto K and it is said to be a nonexpansive retract of E if there exists a nonexpansive retraction of E onto K. If $E = H$, the metric projection P_K is a sunny nonexpansive retraction from H to any closed and convex subset of H. But this is not true in a general Banach spaces. We note from the given lemma (see e.g. [5, 7]) that

Lemma 2.3. *If E is smooth Banach space, C be a convex subset of E, $C_0 \subset C$ and Q is retraction of C onto C_0, then Q is sunny and nonexpansive if and only if for each $x \in C$ and $z \in C_0$ we have $\langle Qx - x, J(Qx - z)\rangle \leq 0$.*

The given theorem known as Reich's Theorem explains the construction of nonexpansive retraction as:

Theorem 2.4 ([13])**.** *Let E be a Banach space which is both uniformly convex and uniformly smooth. Let $T : D(T) \subset E \to E$ be m-accretive. Then for each $x \in E$ the strong $\lim\limits_{r\to 0} J_r(x)$ exists, where $J_r = (I + rT)^{-1}$. Denote the strong $\lim\limits_{r\to 0} J_r(x)$ by Qx; then $Q : E \to \overline{D(T)}$ is a nonexpansive retraction of E onto $\overline{D(T)}$; where $\overline{D(T)}$ denotes the closure of $D(T)$.*

It is well known that under the hypothesis of the above Theorem, $\overline{D(T)}$ is convex.

Lemma 2.5 ([11])**.** *Let E be a real normed linear space. Then the following inequality holds: For each $x, y \in E$,*

$$\|x + y\|^2 \leq \|x\|^2 + 2\langle y, j(x + y)\rangle, \ \forall\ j(x + y) \in J(x + y).$$

Lemma 2.6 ([1] Resolvent Identity)**.** *For $\lambda > 0$ and $\mu > 0$ and $x \in E$, $J_\lambda x = J_\mu \left(\frac{\mu}{\lambda} x + (1 - \frac{\mu}{\lambda}) J_\lambda x\right)$.*

3. Main Result

Theorem 3.1. *Let E be both uniformly smooth and uniformly convex Banach space and $A : D(A) \subset E \to E$ be an m-accretive operator with closed domain and $A^{-1}(0) \neq \phi$. Suppose $\{\alpha_n\}$ be a sequence in $[0,1]$ and $\{r_n\}$ in $(0,\infty)$ such that $\lim_{n\to\infty} \alpha_n = 0$ and $\inf r_n > 0$. Let $x_0 \in D(A)$ be chosen arbitrarily and $\{x_n\}$ be generated by the algorithm*

$$\begin{cases} y_n & = \alpha_n x_0 + (1-\alpha_n) J_{r_n} x_n \\ C_n & = \{z \in E : \|y_n - z\|^2 \leq \|x_n - z\|^2 + 2\alpha_n \langle x_0 - z, j(y_n - z)\rangle\} \\ H_n & = \{z \in D(A) : \langle x_n - x_0, j(x_n - z)\rangle \leq 0\} \\ x_{n+1} & = Q_{C_n \bigcap H_n} x_0 \end{cases} \tag{4}$$

where $S = A^{-1}(0) = F(J_{r_n})$ and Q is the retraction of E onto $\overline{D(A)}$. Then $\{x_n\}$ converges strongly to $Q_S x_0$.

Proof. **(A) $\{x_n\}$ is well defined :** We note here that $S = A^{-1}(0)$ is nonempty, closed and convex. Obviously H_n, $\forall\, n \in \mathbb{N}$ is closed convex, so we first show that C_n is closed and convex for all $n \in \mathbb{N}$. Infact, the inequality in C_n is equivalent to

$$\|x_n - y_n\|^2 + 2\alpha_n \langle x_0 - z, j(y_n - z)\rangle \geq 0.$$

Thus C_n is closed, convex for all $n \in \mathbb{N}$. Next we show that $S = A^{-1}0 = F(J_{r_n})$ is a subset of C_n. For this, let $p \in S$, so that

$$\begin{aligned} \|y_n - p\|^2 &= \|\alpha_n(x_0 - p) + (1-\alpha_n)(J_{r_n} x_n - p)\|^2 \\ &\leq (1-\alpha_n)^2 \|J_{r_n} x_n - p\|^2 + 2\alpha_n \langle x_0 - p, j(y_n - p)\rangle \\ &\leq \|x_n - p\|^2 + 2\alpha_n \langle x_0 - p, j(y_n - p)\rangle, \end{aligned}$$

this implies that $p \in C_n$ and hence $S \subset C_n, \forall\, n \geq 0$. We now show that $S \subset H_n, \forall\, n \geq 0$. For n=0, $S \subset D(A) = H_0$. Let $S \subset H_n$. Since x_{n+1} is the retraction of x_0 onto $C_n \bigcap H_n$, so

$$\langle x_0 - x_{n+1}, j(x_{n+1} - z)\rangle \geq 0,\ \forall\, z \in C_n \bigcap H_n.$$

As $S \subset C_n \bigcap H_n$ by the induction assumption, the last inequality holds, in particular, $\forall\, z \in S$. This together with the definition of H_{n+1} implies that $S \subset H_{n+1}$. Hence $S \subset H_n, \forall\, n \geq 0$. Thus $\{x_n\}$is well defined.

(B) $\{x_n\}$ is bounded: Since $x_n = Q_{H_n} x_0$ (by definition of H_n), so $\langle x_0 - x_n, j(x_n - v)\rangle \geq 0,\ \forall\, v \in H_n$, and since $S \subset H_n$, so we have $\langle x_0 - x_n, j(x_n - p)\rangle \geq 0,\ \forall p \in S$, or

$$\begin{aligned} 0 \leq \langle x_0 - x_n, j(x_n - p)\rangle &= \langle x_0 - p - x_n + p, j(x_n - p)\rangle \\ &\leq \|x_0 - p\|\|x_n - p\| - \|x_n - p\|^2 \end{aligned}$$

i.e. $\|x_n - p\| \leq \|x_0 - p\|,\ \ \forall\, p \in S$. In particular, $\{x_n\}$ is bounded and $\|x_n - q\| \leq \|q - x_0\|,\ \forall\, q \in Q_S(x_0)$.

(C) $\|x_{n+1} - x_n\| \to 0$: From $x_n = Q_{C_n} x_0$ and $x_{n+1} = Q_{C_n \bigcap H_n}(x_0) \subset H_n$ asserts that $\langle x_n - x_0, j(x_n - x_{n+1})\rangle \leq 0$.

$$\begin{aligned} \|x_{n+1} - x_n\|^2 &= \|(x_{n+1} - x_0) + (x_0 - x_n)\|^2 \\ &\leq \|x_{n+1} - x_0\|^2 + 2\langle x_0 - x_n, j(x_{n+1} - x_n)\rangle \\ &= \|x_{n+1} - x_0\|^2 + 2\langle x_n - x_0, j(x_n - x_{n+1})\rangle \\ &\leq \|x_{n+1} - x_0\|^2, \end{aligned}$$

which implies that

$$\|x_{n+1} - x_n\| \to 0 \text{ as } n \to \infty. \tag{5}$$

Thus $\{x_n\}$ is a Cauchy sequence in S and since E is Banach space and S is closed, convex, so there exists $p \in S$ such that $\lim_{n\to\infty} x_n = p$.

(D) $\|x_n - J_r x_n\| \to 0$**:** On the other hand, $x_{n+1} \in C_n$ implies that

$$\|y_n - x_{n+1}\|^2 \leq \|x_n - x_{n+1}\|^2 + 2\alpha_n \langle x_0 - x_{n+1}, j(y_n - x_{n+1})\rangle.$$

Since $\alpha_n \to 0$ as $n \to \infty$, thus using (5) with $\{x_n\}$ bounded, we get that

$$\|y_n - x_{n+1}\| \to 0. \tag{6}$$

Moreover,

$$\|y_n - J_{r_n} x_n\| = \alpha_n \|x_0 - J_{r_n} x_n\| \to 0. \tag{7}$$

So combining (5)- (7) yields

$$\begin{aligned} \|x_n - J_{r_n} x_n\| &= \|x_n - x_{n+1} - y_n + x_{n+1} - J_{r_n} x_n + y_n\| \\ &\leq \|x_n - x_{n+1}\| + \|y_n - J_{r_n} x_n\| + \|x_{n+1} - y_n\| \\ &\to 0. \end{aligned} \tag{8}$$

Putting $r = \inf_{n \geq 0} r_n > 0$ and using Lemma 2.6, we get

$$\begin{aligned} \|J_{r_n} x_n - J_r x_n\| &= \|J_r \left(\frac{r}{r_n} x_n + (1 - \frac{r}{r_n}) J_{r_n} x_n \right) - J_r x_n\| \\ &\leq (1 - \frac{r}{r_n}) \|x_n - J_{r_n} x_n\| \\ &\leq \|x_n - J_{r_n} x_n\|. \end{aligned}$$

Therefore, we have

$$\begin{aligned} \|x_n - J_r x_n\| &\leq \|x_n - J_{r_n} x_n\| + \|J_{r_n} x_n - J_r x_n\| \\ &\leq \|J_{r_n} x_n - x_n\| + \|J_{r_n} x_n - x_n\| \\ &\leq 2\|J_{r_n} x_n - x_n\|, \end{aligned}$$

which implies by (8) that, $\|x_n - J_r x_n\| \to 0$. This together with $\lim_{n\to\infty} x_n = p$ implies that $J_r p = p$, i.e., p is a fixed point of J_r or $p \in S$. Now we claim that $p = Q_S(x_0)$. Since $x_n \in Q_{C_n \bigcap H_n}(x_0)$, so $\langle x_0 - x_n, j(x_n - w)\rangle \geq 0,\ \forall\ w \in S \subset C_n$. Taking limit $n \to \infty$, $\langle x_0 - p, j(p - w)\rangle \geq 0,\ \forall\ w \in S \subset C_n$ or $\langle x_0 - p, j(w - p)\rangle \leq 0$, which implies that $p = Q_S(x_0)$. □

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