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Generalizations of N-Injective Modules

Seminar Paper*

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Abstract: Any left R-module M is said to be p-injective if for every principal left ideal I of R and any R-homomorphism $g : I \to M$, there exists $y \in M$ such that $g(b) = by$ for all b in I. We find that RM is p-injective iff for each $r \in R$, $x \in M$ if $x \notin rM$ then there exists $c \in R$ with $cr = 0$ and $cx \neq 0$. A ring R is said to be epp-ring if every projective R-module is p-injective. Any ring R is right epp-ring iff the trace of projective right R-module on itself is p-injective. A left epp-ring which is not right epp-ring has been constructed.

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1. Introduction

Any ring R is said to be QF if every injective left R-module is projective. Villamayor [3] characterises a ring R over which every simple R-module is injective Faith call it V-ring. R.R Colby defines a ring R as a left (right) IF-ring if every injective left (right) R-module is flat. R.Y.C. Ming [4] characterises the rings over which every simple left R-module is p-injective. Motivated by these ideas here we define a ring R as a left epp-ring if every left projective R-module is p-injective. Through out, R denotes an associative ring with identity and R-modules are unitary.

Definition 1.1. *A left R-module M is called p-injective if for any principal left ideal I of R and any left R-homomorphism $g : I \to M$, there exist $y \in M$ such that $g(b) = by$ for all b in I.*

Definition 1.2. *A left R-module M is f-injective if, for any finitely generated left ideal I of R and any left R- homomorphism $g : I \to M$, there exists $y \in M$ such that $g(b) = by$ for all b in I.*

Definition 1.3. *A ring R is right (left) epp-ring if every right (left) projective R-module is p-injective. A ring R is epp-ring if it is both right as well as left epp-ring. The trace of an R-module M on M is denoted by $T_M(M) = \{Imf | f \in S = End_R M\}$.*

Proposition 1.4.

(i) *Direct product of p-injective modules is p-injective if and only if each factor is p-injective.*

(ii) *Direct sum of p-injective modules is p-injective if and only if each summand is p-injective [4].*

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Proof.

(i) Let the direct product $\prod_{A} M_i$ be p-injective module, to show each M_i is p-injective. For this consider the homomorphism $f_i : I \to M_i$ where $I = (a)$ be any principal left ideal of R generated by 'a'. If $j_i : M_i \to \prod_{A} M_i$ be the inclusion homomorphism then $j_i \circ f_i : I \to \prod_{A} M_i$ is a homomorphism. Since $\prod_{A} M_i$ is p-injective, there exists $(m_i)_{i\in A} \in \Pi M_i$ such that $j_i \circ f_i(x) = x(m_i)_{i\in A}$ for all $x \in I$. Let $q_i : \prod_{A} M_i \to M_i$ be the projection homomorphism. Then $q_i \circ j_i \circ f_i = f_i$ and $q_i \circ j_i \circ f_i(a) = q_i(a(m_i)_{i\in A}) = am_i = f_i(a)$. Hence M_i is p-injective.

Conversely let each M_i is p-injective and $f : I \to \prod_{A} M_i$ be any R-homomorphism. Consider the homomorphism $q_i \circ f : I \to M_i$. By p-injectivity of M_i there exists m_i for each $i \in A$ such that $q_i \circ f(a) = am_i$. Thus $f : I \to \prod_{A} M_i$ is given by $f(a) = a(m_i)_{i\in A}$. Hence $\prod_{A} M_i$ is p-injective.

(ii) Its proof is very much similar to (i).

□

Corollary 1.5. *Direct sum of injective module is p-injective.*

Theorem 1.6. *Following conditions are equivalent for a left R-module M;*

(i) M is left p-injective module.

(ii) For each $r \in R$, $x \in M$ of $x \notin rM$ then there exists $c \in R$ with $cr = 0$ and $cx \neq 0$.

Proof. (i) ⇒ (ii) Suppose (ii) is not true that is for each $r \in R$, $x \in M$ if $x \notin rM$ for some c, $cr = 0$ and $cx = 0$. Put $I = (r) = Rr$ then there is an R-homomorphism $f : I \to M$ defined by $f(r) = x$. By p-injectivity of RM there exists $x' \in M$ such that $f(r) = rx'$ for all $r \in I$. Therefore $x = f(r) = rx' \in rm$ which is a contradiction to the fact that $x \notin rM$.
(ii) ⇒ (i) Let $I = (r) = Rr$ be any principal left ideal and $f : I \to M$ be any R-homomorphism. Suppose $f(r) \neq ry$ for some $y \in M$ and $cr = 0$. This implies that $f(r) \notin rm$ and $f(cr) = cf(r) = 0$ which is a contradiction to the fact that $cr = 0$ and $cx \neq 0$. □

Lemma 1.7 ([5])**.**

(i) Let $A \in |M_R$. The map $\varphi'(A) : Hom(M, A) \otimes {}_S M \to T_M(A)$ is an isomorphism if and only if M generates all kernels of homomorphism $M^n \to A$, $n \in N$.

(ii) The left S-module ${}_S M$ is flat if and only if generates all kernels of homomorphism $M^n \to M$, $n \in N$.

Theorem 1.8. *Following conditions are equivalent for a ring R;*

(i) R is right epp-ring.

(ii) The trace of projective module M_R on itself is p-injective.

(iii) The trace of right free R-module on itself is p-injective.

Proof.
(i) ⇒ (ii) Let R be any right epp-ring i.e. every right projective R-module is p-injective and M be any projective right R-module. Using the fact that every projective is flat and the tensor product of two flat modules is flat we get ${}_S M \otimes_R R \cong {}_S M$ is flat. Therefore by Lemma 1.7 M generates all kernels of the homomorphisms $M^n \to M$, $n \in N$, which implies that

$\varphi'(M) : Hom_R(M, M) \otimes M \to T_M(M)$ is an isomorphism by Lemma 1.7 that is $S \otimes M_R$ is an isomorphic to $T_M(M)$ or M_R is isomorphic to $T_M(M)$ and so $T_M(M)$ is p-injective as R is epp-ring.

(ii) $\Rightarrow$ (iii) obvious.

(iii) $\Rightarrow$ (i) Assume that for every free right R-module M, $T_M(M)$ is p-injective. M would be flat too (free $\Rightarrow$ projective $\Rightarrow$ flat). Using the same arguments as in (i) $\Rightarrow$ (ii) we get $T_M(M)$ is isomorphic to M therefore M is p-injective. Let K be any projective R-module than K would be direct summand of some free R-module say M but M is p-injective. Since direct summand of p-injective is p-injective [proposition 3(ii)] hence K is p-injective that is R is a epp-ring.

This theorem is true for a left epp-ring and left projective R-module M too. □

Example 1.9 ([2, R.R. Colby Example 1]). *A commutative epp-ring which is not a epp-ring modulo its radical. Let* $R = Z \oplus \frac{Q}{Z}$ *with multiplication defined by* $(n_1, q_1)(n_2, q_2) = (n_1 n_2, n_1 q_2 + n_2 q_1), n_i \in Z,\ q_i \in Q$. *Then R is a commutative coherent ring with Jacobson radical*

$$J(R) = \left\{ \frac{(n,\ q)}{\mathrm{n}} = 0 \right\} = \left(0, \frac{Q}{Z} \right)$$

It is obvious that each finitely generated ideal is principal. Thus R is epp-ring but $\frac{\mathrm{R}}{\mathrm{J(R)}} \cong Z$ *which is not a epp-ring as the homomorphism* $f : Z \to Z$ *defined by* $f(nx) = x$ *can not be extended to a homomorphism from* $Z \to Z$.

Example 1.10. *A left epp-ring which is not a right epp-ring [2, Example 2]. Let R be an algebra over a field F with basis* $\{1, e_0, e_1, \ldots, x_1, x_2, \ldots\}$ *for all i, j*

$$e_i e_j = \delta_{i,j} e_j$$

$$x_i e_j = \delta_{i,j+1} x_i$$

$$e_i x_j = \delta_{ij} x_j$$

$$x_i x_j = 0$$

It can be easily verified that R is left coherent and every R-homomorphism $f : {}_R I \to {}_R R$ *extends from* ${}_R R \to {}_R R$. *Thus* ${}_R R$ *is p-injective that is R is left epp-ring. However R is not right epp-ring, since the homomorphism* $x_1 R \to e_0 R$ *via* $x_1 r \to e_0 r$ *can not be extended over R.*

Proposition 1.11. *Let R and S are Marita equivalent ring then R is epp-ring if and only if S is epp-ring.*

Proof. Let F: ${}_R|$M $\to$ ${}_S|$M and G: ${}_S|$M $\to$ ${}_R|$M are category equivalences where ${}_R|$M and ${}_S|$M denote the categories of left R-module and left S-module respectively. Since projectivity is a categorical property we have to show only that M is p-injective in ${}_R|$M if and only if F (M) is p-injective in ${}_S|$M. The sequence with principal ideal I, 0$\to$ I $\to$ R $\to$ $\frac{\mathrm{R}}{\mathrm{I}}$$\to$ 0 is exact in ${}_S|$M if and only if 0$\to$ G (I) $\to$ G(R) $\to$ G ($\frac{\mathrm{R}}{\mathrm{I}}$) $\to$ 0 is exact in ${}_R|$M [1, page 224]. And the sequence 0$\to$ Hom ${}_R$(G ($\frac{\mathrm{R}}{\mathrm{I}}$), M) $\to$ Hom ${}_R$(G (R), M) $\to$ Hom ${}_R$(G (I), M) $\to$ 0 is exact in ${}_R|$M if and only if 0$\to$ Hom$_S$ ($\frac{\mathrm{R}}{\mathrm{I}}$, F(M)) $\to$ Hom$_S$(R, F(M)) $\to$ Hom$_S$(I, F(M)) $\to$ 0 is an exact sequence in ${}_S|$M. That is if M is p-injective in ${}_R|$M if and only if F(M) is p-injective in ${}_S|$M. □

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