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Existence of Common Fixed Points under Compatible Maps

Seminar Paper*

Dr.Shabnam Khan[1]

1 Department of Mathematics, Government Digvijay Autonomous P.G. College, Rajnandgaon (C.G.), India.

Abstract: In this paper we have studied existence of common fixed points under compatible maps. Meir and Keeler established a fixed point theorem for self-mapping f of a metric space. Maiti and Pal generalized this mapping. We will prove our result by using Jungck lemma. We will genralise and improve several results of Meir and Keeler type under commuting and weakly commuting pair of mappings and uniqueness of Common Fixed Point.

Keywords: Existence of common fixed points, Compatible maps, Self-mapping.

1. Introduction

Meir & Keeler established a fixed point theorem for a self-mapping f of a metric space (X, d) satisfying the following condition: For every $\varepsilon > 0$, there exists a $\delta > 0$ such that

$$\varepsilon < d(x,y) < \varepsilon + \delta \Rightarrow d(fx, fy) < \varepsilon \tag{1}$$

Maiti & Pal generalized (1). A self mapping of a metric space (X, d) satisfying the following Condition. For every $\varepsilon > 0$, there exists a $\delta > 0$ such that

$$\varepsilon < \max\{d(x,y), d(x,fx), d(y,fy)\} < \varepsilon + \delta \Rightarrow d((fx),(fy)) < \varepsilon \tag{2}$$

In [4] Park - Rhoades & Rao-Rao generalized Self mapping f & g of metric space (X , d) satisfying the following conditions. For every $\varepsilon > 0$, there exists a $\delta > 0$ such that

$$\varepsilon \leq \max\left\{d(fx,fy), d(fx,gx), d(fy,gy), \frac{1}{2}[d(fx,gy)+d(fy,gx)]\right\} < \varepsilon + \delta \Rightarrow d(gx,gy) < \varepsilon \tag{3}$$

Now we prove a common fixed point theorem of Meir-Keeler type mapping.

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2. Definition and Lemmas

Before presenting main theorem, we give some definitions & lemmas.

Definition 2.1. *let S & T be self mappings of a metric space (X, d). S & T are said to be compatible if* $\lim_{n\to\infty} d(ST(x_n), TS(x_n)) = 0$. *Whenever $\{x_n\}$ is a sequence in x, such that* $\lim_{n\to\infty} S(x_n) = \lim_{n\to\infty} T(x_n) = t$ *for some t in x.*

Definition 2.2. *Let (X, d) be a metric space E, F, S & T are self mappings of X , then a pair $\{E, F\}$ is called a generalized (ε, δ)-$\{S, T\}$- contraction relative to S & T if*

$$E(x) \subset T(x) \ \text{ and } \ F(x) \subset S(x) \tag{4}$$

For every $\varepsilon > 0$, there exists a $\delta > 0$ such that

$$\varepsilon < \max\left\{d(Sx, Ty), d(Sx, Ex), d(Ty, Fy), \frac{1}{2}[d(Sx, Fy) + d(Ty, Ex)]\right\} < \varepsilon + \delta \Rightarrow d(Ex, Fy) < \varepsilon \tag{5}$$

Definition 2.3. *Let E, F, S & T be mappings of a metric space (x, d) into itself such that $E(x) \subset T(x)$ and $F(x) \subset S(x)$. For $x_0 \in X$ any sequence $\{y_n\}$ defined by*

$$y_{2n-1} = Tx_{2n-1} = Ex_{2n-2} \tag{6}$$

$$y_{2n} = Sx_{2n} = Fx_{2n-1}. \tag{7}$$

For $n \in N$, is called an $\{S, T\}$ iterate of x_0.

The following Lemma of Jungck is used to prove our main results:

Lemma 2.4. *Let $S, T : (x, d) \to (x, d)$ be mappings. Let S and T be compatible and $S(x_n)$, $T(x_n) \to t$ for some $t \in X$. Then we have* $\lim_{n\to\infty} TS(x_n) = S(t)$, *if S is continuous.*

Lemma 2.5. *Let (x, d) be a metric space and pair $\{E, F\}$ be a generalized (ε, δ)-$\{S, T\}$- contraction. If $x_0 \in X$ and $\{y_n\}$ is an S, T-iteration of x_0 under E & F, then for each $\varepsilon > 0$ there exists $\delta > 0$ such that $\varepsilon \le d(y_p, y_q) < \varepsilon + \delta \Rightarrow d(y_{p+1}, y_{q+1}) < \varepsilon$ Where p & q are opposite parts.*

Proof. Since E & F is a generalized $(\varepsilon - \delta) - \{S, T\}$-contraction

$$\varepsilon \le \max\left\{d(Sx, Ty), d(Sx, Ex), d(Ty, Fy), \frac{1}{2}[d(Sx, Fy) + d(Ty, Ex)]\right\} < \varepsilon + \delta \Rightarrow d(Ex, Fy) < \varepsilon. \tag{8}$$

Suppose $\varepsilon < d(Yp, Yq) < \varepsilon + \delta$. Putting $p = 2n$ & $q = 2m - 1$ in (8)

$$\begin{aligned} d(y_{p+1}, y_{q+1}) &= d(y_{2n+1}, y_{2m}) \\ &= d(Ex_{2n}, Fx_{2m-1}) \ \text{ and} \\ d(y_p, y_q) &= d(Y_{2n}, Y_{2m.1}) \\ &= d(Sx_{2n}, Tx_{2m-1}) \\ &< \max\left\{d(Sx_{2n}, Tx_{2m-1}), d(Sx_{2n}, Ex), d(Tx_{2m-1}, Fx_{2m-1}), \frac{1}{2}[d(SX_{2n}, Fx_{2m-1}) + d(Tx_{2m-1}, EX_{2n})]\right\} \\ &\quad + d(Tx_{2m-1}, Ex_{2n}) \end{aligned}$$

Hence from (8) $\varepsilon \le d(Yp, Yq) < \varepsilon + \delta \Rightarrow d(Y_{p+1}, Y_{q+1}) < \varepsilon$ This completes the proof . □

Lemma 2.6. *let E, F, S & T are self mapping of a metric space (x, d) satisfying the hypothesis of the Lemma 2.4, then* $\lim_{n\to\infty} d(y_n, y_{n+1}) = 0$.

Proof. Let x_0 be any point in X. By (8), we have

$$\begin{aligned} d(y_{2n}, y_{2n+1}) &= d(Ex_{2n}, Ex_{2n-1}) \\ &< \max\left\{d(Sx_{2n}, Tx_{2n-1}), d(Sx_{2n}, Ex_{2n}), d(Tx_{2n-1}, Fx_{2n-1}), \frac{1}{2}[d(Sx_{2n}, Fx_{2n-1}) + d(Tx_{2n-1}, Ex_{2n})]\right\} \\ &= \max\left\{d(y_{2n}, y_{2n-1}), d(y_{2n}, y_{2n+1}), d(y_{2n-1}, y_{2n}), \frac{1}{2}[0 + d(y_{2n-1}, y_{2n+1})]\right\} = d(y_{2n-1}, y_{2n}) \end{aligned}$$

Or Equivalently $d(y_{2n}, y_{2n+1}) < d(y_{2n-1}, y_{2n})$. Similarly, we have $d(y_{2n+1}, y_{2n+2}) < d(y_{2n}, y_{2n+1})$. Thus the sequence $\{d(y_n, y_{n+1})\}$ is a monotone decreasing sequence & it converges to its greatest lower bound of its range $t \geq 0$. In fact $t = 0$ otherwise there exists $\delta > 0$ and for some $n \geq N$. Lemma J $\Rightarrow t \leq d(y_m, y_{m+1}) < t + \delta$. Put $m = 2n$ and note that

$$\max\left\{d(Sx_{2n}, Tx_{2n+1}), d(Sx_{2n}, Ex_{2n}), d(Tx_{2n+1}, Fx_{2n+1}), \frac{1}{2}[d(Sx_{2n}, Fx_{2n+1}) + d(Tx_{2n}, Ex_{2n})]\right\}$$

$$\begin{aligned} &= \max\{d(y_{2n}, y_{2n+1}), d(y_{2n}, y_{2n+1}), d(y_{2n+1}, y_{2n+2}), \frac{1}{2}[d(y_{2n}, y_{2n+2}) + 0]\} \\ &= d(y_{2n}, y_{2n+1}). \end{aligned}$$

Since $\frac{1}{2}d(y_{2n}, y_{2n+2}) < \frac{1}{2}[d(y_{2n}, y_{2n+1}) + d(y_{2n+1}, y_{2n+2})] < d(y_{2n}, y_{2n+1})$. By using Lemma 2.5, $t \leq d(y_m, y_{m+1}) < t + \delta \Rightarrow d(y_{m+1}, y_{m+2}) < t$. But $d(y_{m+1}, y_{m+2}) = d(Tx_{2n+1}, Sx_{2n+2}) < t$. Which is a contradiction. Therefore we have $\lim_{n\to\infty} d(y_n, y_{n+1}) = 0$. □

Lemma 2.7. *Let the mapping E, F, S & T are as in Lemma 2.6, then the indicated sequence $\{Y_n\}$ is a Cauchy sequence. Theorem 3 extends, generalizes & improves several results of Meir & Keeler type under commuting & weakly commuting pair of mappings.*

Theorem 2.8. *Let E, F, S, and T be mappings of a complete metric space (X, d) into itself satisfying Definition 2.2, 2.3, (8) E, S and F, T are compatible pairs, one of E, F, S and T is continuous. Then E, F, S and T have a unique Common fixed point in X.*

Proof. Since (X, d) is a complete metric space & using Lemma 2.7, the sequence $\{y_n\}$ is a Cauchy sequence and hence it is a convergent sequence, call the limit z in X. Since $\{Ex_{2n}\}$, $\{Fx_{2n-1}\}$, $\{Sx_{2n}\}$ and $\{Tx_{2n-1}\}$ are subsequence of $\{y_n\}$ so there are also converge to z in X. Now, suppose S is continuous and $\{E, S\}$ is a compatible Pair so by Lemma 2.5 we have SEx_{2n}, SSx_{2n}, $ESx_{2n} \Rightarrow S_z$. Now, we claim that $Sz = z$. If $S_z \neq z$, then we have

$$\lim_{n\to\infty} \max\left\{d(SSx_{2n}, Tx_{2n+1}), d(SSx_{2n}, ESx_{2n}), d(Tx_{2n+1}, Fx_{2n+1}), \frac{1}{2}[d(SSx_{2n}, Fx_{2n+1}) + d(Tx_{2n+1}, ESx_{2n}]\right\} = d(S_z, z)$$

Choose $\varepsilon = \frac{1}{2}d(S_z, z)$, then there exists a positive integer N, such that $n \geq N$,

$$\max\left\{d(SSx_{2n}, Tx_{2n+1}), d(SSx_{2n}, ESx_{2n})d(Tx_{2n+1}, Fx_{2n+1}), \frac{1}{2}[d(SSx_{2n}, Fx_{2n+1}) + d(Tx_{2n+1}, ESx_{2n})]\right\} - \{d(S_z, z)\} < \varepsilon$$

That is

$$\begin{aligned} &\varepsilon = d(S_z, z) - \varepsilon \\ &< \max\left\{d(SSx_{2n}, Tx_{2n+1}), d(SSx_{2n}, ESx_{2n}), d(Tx_{2n+1}, Fx_{2n+1}), \frac{1}{2}[d(SSx_{2n}, Fx_{2n+1}) + d(Tx_{2n+1}, ESx_{2n})]\right\} < d(S_Z, z) + \varepsilon. \end{aligned}$$

Since $\lim_{n\to\infty} d(ESx_{2n}, Fx_{2n+1}) = d(S_z, z)$, there exists $N_2 > N_1$ such that for all $n > N_2$, $|d(ESx_{2n}, Fx_{2n+1}) - d(S_z, z)| < \frac{\varepsilon}{2}$. That is $\frac{-\varepsilon}{2} + d(S_z, z) < d(ESx_{2n}, Fx_{2n+1}) < \varepsilon$ implies $d(S_z, z) < \varepsilon$, a contradiction. Hence, $S_z = z$. Now, again we claim that $Ez \neq z$. For if $Ez \neq z$, then we have

$$\lim_{n\to\infty} \max\left\{d(Sz, Tx_{2n+1}), d(Sz, Ez), d(Tx_{2n+1}, Fx_{2n+1}), \frac{1}{2}[d(Sz, Fx_{2n+1}) + d(Tx_{2n+1}, Ez)]\right\} = d(Ez, z).$$

Choose $\varepsilon = \frac{1}{2}d(Ez, z)$. Then there exists $M_1 > 0$ such that for all $n \leq M_1$,

$$\max\left\{d(Sz, Tx_{2n+1}), d(Sz, Ez), d(Tx_{2n+1}, Fx_{2n+1}), \frac{1}{2}[d(Sz, Fx_{2n+1}) + d(Tx_{2n+1}, Ez)]\right\} - d(Ez, z) < \varepsilon'$$

That is, $\varepsilon' = \varepsilon' + d(Ez, Z) < \max\left\{d(Sz, Tx_{2n+1}), d(Sz, Ez), d(Tx_{2n+1}, Fx_{2n+1}), \frac{1}{2}[d(Sz, Fx_{2n+1}) + d(Tx_{2n+1}, Ez)]\right\} < d(Ez, z) + \varepsilon'$. Since $\lim_{n\to\infty} d(Ez, Fx_{2n+1}) = d(Ez, z)$ there exists $M_2 > M_1$ such that for all $n \geq M_2$, $|d(Ez, Fx_{2n+1}) - d(Ez, z)| < \frac{\varepsilon}{2}$. That is $-\frac{\varepsilon'}{2} + d(Ez, z) < d(Ez, Fx_{2n+1}) < \varepsilon' = \frac{1}{2}d(Ez, z)$ implies $d(Ez, z) < \varepsilon'$, a contradiction. Thus we have $Ez = z$. Since $E(x) \subset T(x)$, there exists $u \in X$ such that $Z = Sz = Ez = Tu$. Further, we claim that $z = Fu$. Otherwise

$$d(z, Fu) = d(Ez, Fu) < \max\left\{d(Sz, Tu), d(Sz.Ez), d(Tu, Fu), \frac{1}{2}[d(Sz, Fu) + d(Tu, Ez)]\right\} = d(z, Fu)$$

Which is contraction and hence $Z = Fu = Tu$. Since, F & T are compatible maps. So, that $d(TFu, FTu) = 0 \Rightarrow TFu = FTu \Rightarrow Tz = Fz$. Now, we claim that $Fz = z$. If $Fz \neq z$, then we have

$$d(z, Fz) = (Ez, Fz) < \max\left\{d(Sz, Tz), d(Sz, Ez), d(Tz, Fz), \frac{1}{2}[d(Sz, Fz) + d(Tz, Ez)]\right\} = d(Fz, z)$$

A contradiction. Then we have, $z = Fz = Tz$. Therefore, z is a common fixed point of E, F, S & T.

Uniqueness of common fixed point z.

Suppose z^* be a second common fixed point of E, F, S and T. Then again by using, we have

$$d(z, z^*) = d(Ez, Fz^*) < \max\left\{d(Sz, Tz*), d(Sz, Ez), d(Tz*, Fz*), \frac{1}{2}[d(Sz, Fz) + d(Tz*, Ez)]\right\} = d(z, z*)$$

Which is contradiction. Hence, z is a unique common fixed point of E, F, S and T. Similarly, we can also complete the proof if T or E or F is continuous. This completes the proof. □

References

[1] G.Jungck, *Compatible mappings and common fixed points*, Int. J. Math. & Math. Sci., 9(1986), 771-779.

[2] G.Jungck and B.E.Rhoades, *Some fixed fixed points theorems for compatible maps*, Int. J. Math. & Math. Sci., 16(1993), 417-428.

[3] S.Park and B.E.Rhoades, *Meir-Keeler type contractive conditions*, Math. Japanica, 26(1)(1981), 13-20.

[4] H.K.Pathak, *A Meir-Keeler type fixed point theorem for weakly uniformly contraction maps*, Bull. Malasian Math soc. (IT series), 13 (1990), 21-29.

[5] H.K.Pathak, *A note on fixed point theorem of Rao-Rao*, Bull. Calcutta Math. Soc., 79 (1987), 267-273.