International Journal of Current Research in Science and Technology
Volume 1, Issue 12 (2015), 1–10.
ISSN: 2394-5745
Available Online: http://ijcrst.in/

International Journal of Current Research in Science and Technology

Some Properties of Bivariate Life Distributions

Research Article

U.Rizwan[1]* and Syed Tahir Hussainy[1]

1 Department of Mathematics, Islamiah College, Vaniyambadi, Tamilnadu, India.

Abstract: In this paper, we introduce some new class of bivariate life distributions and study the closure properties under the Formation of Coherent systems, properties on Partial Ordering, Preservation Properties for some Bivariate Life Distributions.

Keywords: Bivariate life distributions, Coherent Systems, Partial Ordering, Preservation Properties.

1. Introduction

Bivariate notions of aging and their related classes of life distributions defined by aging properties play a central role in survival analysis, reliability theory, maintenance models and many other actuarial science, engineering, economics, biometry and applied probability areas. They are also useful in obtaining fundamental inequalities of estimates. In the last four decades, remarkable studies have been done on the different aspects of univariate life distributions. Recently, studies have been attracted to establish bivariate life distributions.

2. Definitions and Some Related Concepts

In reliability theory, ageing life is usually characterized by a nonnegative random variable $x \geq 0$ with cumulative distribution function (cdf) $F(\cdot)$ and survival function $\overline{F}(\cdot) = 1 - F(\cdot)$. For any random variable X, let

$$X_t \cong [X - t|X > t], \quad t \in \{x : F(x) < 1\}$$

denote a random variable whose distribution is the same as the conditional distribution of $X - t$ given that $X > t$ When X is the lifetime of a device, X_t can be regarded as the residual lifetime of the device at time t, given that the device has survived upto time t. Its survival function is

$$\overline{F}_t(x) = \frac{\overline{F}(t + x)}{\overline{F}(t)}, \overline{F}(t) > 0$$

where $F(x)$ is the survival function of X

Remark 2.1. *If $F(\cdot)$ is an exponential distribution then $\overline{F}_t(x) = \overline{F}(x)$*

* *E-mail: vinayshukla.itmmaharajganj@gmail.com*

Definition 2.2 ([8]). *A bivariate random variable (X, Y) or its distribution $\overline{F}(t, s)$ is said to have Bivariate New Better than Used (BNBU), if*

$$\overline{F}(x+t, y+s) \leq \overline{F}(x, y) \cdot \overline{F}(t, s)$$

for $x, y, t, s \geq 0$.

Definition 2.3 ([8]). *A bivariate random variable (X, Y) or its distribution $\overline{F}(t, s)$ is said to have Bivariate New Better than Used in Expectation (BNBUA), if*

$$\int_0^v \int_0^u \overline{F}(x+t, y+s)\, dt\, ds \leq \overline{F}(x, y) \int_0^v \int_0^u \overline{F}(t, s) dt ds,$$

for $x, y, t, s \geq 0$ and u, v are finite.

Definition 2.4 ([10]). *A bivariate random variable (X, Y) or its distribution $\overline{F}(t, s)$, having failure rate $r(x, y)$, is said to have Bivariate New Better than Used in Failure Rate Average (BNBUFRA), if*

$$r(0,0) \leq \frac{1}{\sqrt{t^2+s^2}} \int_0^t \int_0^s r(x, y) dy dx\, ; \ \ for\ 0 \leq t < \infty \ and\ 0 \leq s < \infty.$$

Definition 2.5. *Let F and G be two arbitrary BIFR life distributions. We say that F is more BIFR than G, (written as, $F \overset{BIFR}{<} G$,) if $\left(G^{-1} \circ F\right)(x, y)$ is convex. If the failure rate exist, then an equivalent formulation is*

$$\frac{r_F\left(F^{-1}(u, v)\right)}{r_G\left(G^{-1}(u, v)\right)}$$

is nondecreasing in $v \in [0, 1]$. This ordering is scale invariant.

Definition 2.6. *Let $\mathfrak{F}$ be a (family) class of absolutely continuous life distribution. Let F and G be two arbitrary life distributions in $\mathfrak{F}$ and $F(0,0) = 0$, $G(0,0) = 0$ with positive and right continuous densities f and g, respectively. F is convex ordered with respect to G if, and only if $G^{-1}F$ is convex on the interwal, where $0 < F < 1$. Assume that G is always fixed. Let*

$$H_F^{-1}(t, s) = \int_0^{F^{-1}(t)} \int_0^{F^{-1}(s)} g\left[G^{-1}F(u, v)\right] du\, dv; \quad 0 \leq t, s \leq 1.$$

Definition 2.7. *Let F and G be continuous bivariate life distributions and G be increasing on its support and $F(0,0) = 0\ ;\ G(0,0) = 0$, then F is star-shaped with respect to G, (written as, $F \overset{*}{<} G$,) if $\left(G^{-1} \circ F\right)(x, y)$ is Star-shaped. Equivalently, $\frac{\left(G^{-1}F\right)(x,y)}{xy}$ is increasing for $x, y > 0$.*

Definition 2.8. *Let F and G be two arbitrary BNBUE life distributions. We say that F is more BNBUE than G, (written as, $F \overset{BNBUE}{<} G$,) if*

$$\frac{\mu_F\left(F^{-1}(u, v)\right)}{\mu_G\left(G^{-1}(u, v)\right)} \leq \frac{\mu_F}{\mu_G}, \tag{1}$$

where the mean residual life of F is μ_F.

Definition 2.9. *Let F and G be two arbitrary BDMRL life distributions. We say that F is more BDMRL than G, (written as, $F \overset{BDMRL}{<} G$,) if $\frac{\mu_F\left(F^{-1}(u,v)\right)}{\mu_G\left(G^{-1}(u,v)\right)}$ is decreasing in $u \in [0, 1]$ and decreasing in $v \in [0, 1]$.*

Definition 2.10. *Let F and G be two arbitrary BHNBUE life distributions. We say that F is more BHNBUE than G, (written as, $F \overset{BHNBUE}{<} G$,) if $\frac{G_e^{-1} \circ F_e(x,y)}{xy} \geq \frac{\mu_G}{\mu_F}$ for all $x, y \geq 0$.*

Definition 2.11. *Let F and G be two arbitrary BNBUFR life distributions. We say that F is more BNBUFR than G, (written as, $F \overset{BNBUFR}{<} G$,) if $\alpha'(x, y) \geq \alpha'(0, 0)$, where $\alpha(x, y) = \left(G^{-1} \circ F\right)(x, y) = \left(\overline{G}^{-1} \circ \overline{F}\right)(x, y)$.*

3. Formation of Coherent systems

It is often of interest in reliability applications to see if life distribution classes are preserved under the reliability operations. This section studies the question whether the classes of Bivariate Life distributions under the operations of formation of coherent systems are closed. The results show that, formation of coherent systems are closed.

Theorem 3.1. *Suppose each of the independent components of a coherent system has a BNBU life distribution. Then the system itself is a BNBU life distribution.*

Proof. Let F denote the life distribution of a system, while $F_i : \quad i = 1, 2, \dots, n$ denotes the life distribution of the ith component of the system. Then for $0 < \alpha \leq 1$,

$$\overline{F}(\alpha t_1, \beta t_2) = h\left[\overline{F}_1(\alpha t_1, \beta t_2)\ ,\ \overline{F}_2(\alpha t_1, \beta t_2)\ , \dots, \overline{F}_n(\alpha t_1, \beta t_2)\right]$$

Since, each F_i is BNBU, $\overline{F}_i(x+t, y+s) \leq \overline{F}_i(x,y) \cdot \overline{F}_i(t,s) \ : \quad i = 1, 2, \dots, n$. Also, since h is increasing in each argument, it follows that

$$\overline{F}(\alpha t_1, \beta t_2) \geq h\left[\overline{F}_1^{\sqrt{\alpha^2+\beta^2}}(t_1, t_2)\ ,\ \overline{F}_2^{\sqrt{\alpha^2+\beta^2}}(t_1, t_2)\ , \dots, \overline{F}_n^{\sqrt{\alpha^2+\beta^2}}(t_1, t_2)\right]. \tag{2}$$

Following, Barlow and Proschan (1975), a theorem, we have

$$h\left[\overline{F}_1^{\sqrt{\alpha^2+\beta^2}}(t_1, t_2)\ ,\ \overline{F}_2^{\sqrt{\alpha^2+\beta^2}}(t_1, t_2)\ , \dots, \overline{F}_n^{\sqrt{\alpha^2+\beta^2}}(t_1, t_2)\right] \geq h^{\sqrt{\alpha^2+\beta^2}}\left[\overline{F}_1(t_1, t_2)\ ,\ \overline{F}_2(t_1, t_2)\ , \dots, \overline{F}_n(t_1, t_2)\right],$$

so that the inequality (5.1) becomes

$$\overline{F}(\alpha t_1, \beta t_2) \geq h^{\sqrt{\alpha^2+\beta^2}}\left[\overline{F}_1(t_1, t_2)\ ,\ \overline{F}_2(t_1, t_2)\ , \dots, \overline{F}_n(t_1, t_2)\right]$$

and this completes the proof of the theorem. □

Remark 3.2. *Let $\overline{\mathbf{z}} = (z_1, z_2, \dots, z_n)$ with $z_i = (x_i, y_i)$, where*

$$z_i = \begin{cases} 1 & \text{if the component } i \text{ functions} \\ 0 & \text{if the component } i \text{ fails} \end{cases}$$

for $i = 1, 2, 3, \dots, n$. Then the structure function of the system is

$$\chi(\overline{\mathbf{z}}) = \begin{cases} 1 & \text{if the system functions} \\ 0 & \text{if the system fails} \end{cases}$$

A system is s-coherent if the structure function χ is non-decreasing and non-constant in any x_i. The reliability function $h(\mathbf{p})$ of s-coherent system with structure function χ is defined by $h(\mathbf{p}) \equiv \Pr\{\chi(\overline{\mathbf{z}}) = 1\}$, where the state of the component in a s-coherent system are indicated by s independent random variables x_i with $Pr\{z_i = 1\} = p_i$; $\Pr\{z_i = 0\} = 1 - p_i$, $i = 1, 2, 3, \dots, n$. Let the component i have the survival probability $\overline{F}_i(t,s)$ and the system have $\overline{F}(t,s)$, then $\overline{F}(t,s) = h\left(\overline{F}_1(t,s), \overline{F}_2(t,s), \dots, \overline{F}_n(t,s)\right)$. The failure rate function r_i of the component i is defined by $r_i(t,s) = -\log \overline{F}(t,s)$ and the failure rate of the system is $\mathbf{r}(t,s) = -\log h\left[e^{-r_1(t,s)}, e^{-r_2(t,s)}, \dots, e^{-r_n(t,s)}\right]$. Define the failure rate transform by $\eta(\mathbf{r}) \equiv -\log h\left[e^{-r_1}, e^{-r_2}, \dots, e^{-r_n}\right]$. It may be noted that the failure rate transform defined above is non-decreasing in each argument and super-additive.

Theorem 3.3. *Suppose that each component of a s-coherent system has a bivariate NBUFRA distribution. Then the system itself has a bivariate NBUFRA distribution.*

Proof. Since the failure rate transformation is super-additive, the inequality

$$\frac{1}{\sqrt{t^2+s^2}}\eta(\mathbf{r}(t,s)) \geq \lim_{(u,v)\downarrow(0,0)} \frac{1}{\sqrt{u^2+v^2}}\eta(\mathbf{r}(u,v)), \quad t,s>0$$

is trivial if the right hand side is infinite. Now assume that the right hand side of the inequality is finite. Since, η has continuous partial derivatives, we have

$$\lim_{(u,v)\downarrow(0,0)} \frac{1}{\sqrt{u^2+v^2}}\eta(\mathbf{r}(u,v)) = \sum_{i=1}^{n}\left(\frac{\partial\eta(0,0)}{\partial r_i}\cdot\frac{\partial r_i(0,0)}{\partial t} + \frac{\partial\eta(0,0)}{\partial r_i}\cdot\frac{\partial r_i(0,0)}{\partial s}\right).$$

Since each component has BNBUFRA life distributions,

$$\frac{r_i(t,s)}{\sqrt{t^2+s^2}} \geq \lim_{(u,v)\downarrow(0,0)} \frac{1}{\sqrt{u^2+v^2}} r_i(u,v).$$

By monotonicity and continuity of η, we have

$$\begin{aligned}
\frac{1}{\sqrt{t^2+s^2}}\eta(\mathbf{r}(t,s)) &\geq \lim_{(u,v)\downarrow(0,0)} \frac{1}{\sqrt{t^2+s^2}}\eta\left\{\frac{\sqrt{t^2+s^2}}{\sqrt{u^2+v^2}}\mathbf{r}(u,v)\right\} \\
&\geq \lim_{(u,v)\downarrow(0,0)} \lim_{(s,t)\downarrow(0,0)} \frac{1}{\sqrt{t^2+s^2}}\eta\left\{\frac{\sqrt{t^2+s^2}}{\sqrt{u^2+v^2}}\mathbf{r}(u,v)\right\} \\
&\geq \sum_{i=1}^{n}\left(\frac{\partial\eta(0,0)}{\partial r_i}\cdot\frac{\partial r_i(0,0)}{\partial t} + \frac{\partial\eta(0,0)}{\partial r_i}\cdot\frac{\partial r_i(0,0)}{\partial s}\right) \\
&= \lim_{(u,v)\downarrow(0,0)} \frac{1}{\sqrt{u^2+v^2}}\eta(\mathbf{r}(u,v))
\end{aligned}$$

This completes the proof. □

Consider a system of n independent and not necessarily identical components in which the ith component has the survival function $\overline{F}_i = 1-F_i, i=1,2,\ldots,n$. Let $h(\mathbf{p}) = h(p_1,p_2,\ldots,p_n)$ denotes the system reliability function. In the next theorem, we compare the random life times of two systems based on $\overset{BIFR}{<}$ ordering.

Theorem 3.4. *If $\sum_{i=1}^{n} p_i \frac{\partial h(\mathbf{p})}{\partial p_i}$ is increasing in p_i for all $i=1,2,\ldots,n$. Then $h(X) \overset{BIFR}{\leq} h(Y)$, whenever $X_i \overset{BIFR}{\leq} Y$, $i = 1,2,\ldots,n$.*

Proof. For $z_1, z_2 > 0$ and $t,s \geq 0$, we have

$$r_{h(X)}(z_1+t, z_2+s) = \sum_{i=1}^{n} r_{X_i}(z_1+t,z_2+s)\overline{F}_{X_i}(z_1+t,z_2+s)\frac{1}{h(\mathbf{p})}\frac{\partial h(\mathbf{p})}{\partial p_i}\bigg|_{p_i=\overline{F}_{X_i}(z_1+t,z_2+s)}$$

Since, $X_i \overset{BIFR}{\leq} Y$, by definition, we have

$$\begin{aligned}
r_{h(X)}(z_1+t,z_2+s) &\geq r_Y(z_1+t,z_2+s)\sum_{i=1}^{n}\overline{F}_{X_i}(z_1+t,z_2+s)\frac{1}{h(\mathbf{p})} \\
&\geq r_Y(z_1+t,z_2+s)\sum_{i=1}^{n}\overline{F}_Y(z_1+t,z_2+s)\frac{1}{h(\mathbf{p})}\frac{\partial h(\mathbf{p})}{\partial p_i}\bigg|_{p_i=\overline{F}_{X_i}(z_1+t,z_2+s)} \\
&= r_{h(Y)}(z_1+t,z_2+s).
\end{aligned}$$

This completes the proof. □

Definition 3.5. *Let $Z_1 = (X_1, Y_1)$ and $Z_2 = (X_2, Y_2)$ be the discrete bivariate random variables with survival functions*

$$\overline{F}(k_1, k_2) = Pr(X_1 > k_1\,,\, Y_1 > k_2) \quad and \quad \overline{G}(k_1, k_2) = Pr(X_2 > k_1\,,\, Y_2 > k_2),$$

respectively for $k_1, k_2 \in \mathbb{Z}^+$. The random variable Z_1 is smaller than the random variable Z_2 in generating function order, (written as, $Z_1 \overset{g}{<} Z_2$,) if

$$\sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty} s^{k_1}\, t^{k_2}\overline{F}(k_1, k_2) \ \leq \ \sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty} s^{k_1}\, t^{k_2}\overline{G}(k_1, k_2),$$

for all $0 < s, t < 1$.

Definition 3.6. *A non-negative random variable $Z = (X, Y)$, with survival function $\overline{F}$, is said to be a discrete BNBU (BNWU) in probability generating function (pgf) order, (written as, $F \in$ discrete $BNBU_{pg}$ $(BNWU_{pg})$,) if*

$$\sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty} s^{k_1}\, t^{k_2}\overline{F}(x+k_1, y+k_2) \ \leq (\geq) \ \overline{F}(x, y)\sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty} s^{k_1}\, t^{k_2}\overline{F}(k_1, k_2),$$

for all $x, y \in \mathbb{Z}^+$ and $0 < s, t < 1$.

Theorem 3.7. *Let $Z = (X, Y)$, be a bivariate random variable with distribution function F and let $W = h(Z)$ where $h : \mathbb{Z}^+ \longrightarrow \mathbb{Z}^+$ is increasing and g denote the inverse of h. If Z is a discrete $BNBU_{pg}$ and $g(x)$ is star-shaped, then W is also a discrete $BNBU_{pg}$.*

Proof. Let G denote the distribution of $W = h(Z)$. Then

$$G(k_1, k_2) = G\left(g(k_1), g(k_2)\right).$$

Since Z is a discrete BNBU in probability generating function order

$$\sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty} s^{k_1}\, t^{k_2}\overline{F}(x+k_1, y+k_2) \ \leq \ \overline{F}(x, y)\sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty} s^{k_1}\, t^{k_2}\overline{F}(k_1, k_2),$$

Consider

$$\begin{aligned}
s^{k_1}\, t^{k_2}\overline{G}(k_1, k_2)\sum_{l_1=0}^{\infty}\sum_{l_2=0}^{\infty} s^{l_1}\, t^{l_2}\overline{G}(l_1, l_2) \ - \ \sum_{l_1=k_1}^{\infty}\sum_{l_2=k_2}^{\infty} s^{l_1}\, t^{l_2}\overline{G}(l_1, l_2)\\
= \ s^{k_1}\, t^{k_2}\overline{G}(k_1, k_2)\sum_{l_1=0}^{k_1}\sum_{l_2=0}^{k_1} s^{l_2}\, t^{l_2}\overline{G}(l_1, l_2)\\
- \ \left[1 - s^{k_1}\, t^{k_2}\overline{G}(k_1, k_2)\right]\sum_{l_1=k_1+1}^{\infty}\sum_{l_2=k_2+1}^{\infty} s^{l_1}\, t^{l_2}\overline{G}(l_1, l_2)\\
= \ s^{k_1}\, t^{k_2}\overline{G}(g(k_1), g(k_2))\sum_{l_1=0}^{k_1}\sum_{l_2=0}^{k_1} s^{l_2}\, t^{l_2}\overline{G}(g(l_1), g(l_2))\\
- \ \left[1 - s^{k_1}\, t^{k_2}\overline{G}(g(k_1), g(k_2))\right]\sum_{l_1=k_1+1}^{\infty}\sum_{l_2=k_2+1}^{\infty} s^{l_1}\, t^{l_2}\overline{G}(g(l_1), g(l_2))\\
\geq \ s^{k_1}\, t^{k_2}\overline{G}(g(k_1), g(k_2))\sum_{l_1=0}^{k_1}\sum_{l_2=0}^{k_1} s^{l_2}\, t^{l_2}\overline{G}\left(\frac{g(k_1)}{k_1}\, l_1\,,\, \frac{g(k_2)}{k_2}\, l_2\right)\\
- \ \left(1 - s^{k_1}\, t^{k_2}\overline{G}(g(k_1), g(k_2))\right)\\
\sum_{l_1=k_1+1}^{\infty}\sum_{l_2=k_2+1}^{\infty} s^{l_1}\, t^{l_2}\overline{G}\left(\frac{g(k_1)}{k_1}\, l_1\,,\, \frac{g(k_2)}{k_2}\, l_2\right)
\end{aligned}$$

$$
\begin{aligned}
&= s^{\alpha_1 g(k_1)}\, t^{\alpha_2 g(k_2)} \overline{G}(g(k_1), g(k_2)) \sum_{y_1=0}^{g(k_1)} \sum_{y_2=0}^{g(k_1)} s^{\alpha_1 y_1}\, t^{\alpha_2 l y_2} \overline{G}\left(y_1, y_2\right) \\
&\quad - \left[1 - s^{\alpha_1 g(k_1)}\, t^{\alpha_2 g(k_2)} \overline{G}(g(k_1), g(k_2))\right] \\
&\qquad \sum_{y_1=g(k_1)+1}^{\infty} \sum_{y_2=g(k_2)+1}^{\infty} s^{\alpha_1 y_1}\, t^{\alpha_2 y_2} \overline{G}\left(y_1, y_2\right) \\
&= s^{\alpha_1 g(k_1)}\, t^{\alpha_2 g(k_2)} \overline{G}(g(k_1), g(k_2)) \sum_{y_1=0}^{\infty} \sum_{y_2=0}^{\infty} s^{\alpha_1 y_1}\, t^{\alpha_2 l y_2} \overline{G}\left(y_1, y_2\right) \\
&\quad - \sum_{y_1=g(k_1)+1}^{\infty} \sum_{y_2=g(k_2)+1}^{\infty} s^{\alpha_1 y_1}\, t^{\alpha_2 y_2} \overline{G}\left(y_1, y_2\right),
\end{aligned}
$$

which is positive, since Z is a discrete BNBU$_{pg}$. The inequality is because g is star-shaped and where $y_1 = \frac{g(k_1)}{k_1} l_1$; $y_2 = \frac{g(k_2)}{k_2} l_2$ with $\alpha_1 = \frac{k_1}{g(k_1)}$; $\alpha_2 = \frac{k_2}{g(k_2)}$. It follows that W is also a discrete BNBU$_{pg}$ and the proof is complete. □

We next state the dual cse of the preceding result in the next theorem.

Theorem 3.8. *Let $Z = (X, Y)$, be a bivariate random variable with distribution function F and let $W = h(Z)$ where $h : \mathbb{Z}^+ \longrightarrow \mathbb{Z}^+$ is increasing and g denote the inverse of h. If Z is a discrete BNWU$_{pg}$ and $g(x)$ is star-shaped, then W is also a discrete BNWU$_{pg}$.*

4. Preservation Properties and Partial Ordering for some Bivariate Life Distributions

In this section, we prove some preservation properties and the Properties on Partial Ordering for some Bivariate Life Distributions.

Theorem 4.1. *If $F_1 \overset{C}{<} F_2$, $F_1, F_2, G \in \mathfrak{F}$, gG^{-1} is uniformly continuous on $[0,1]$, then $\frac{H_{F_1}^{-1}(t,s)}{H_{F_1}^{-1}(1,1)} \geq \frac{H_{F_2}^{-1}(t,s)}{H_{F_2}^{-1}(1,1)}$; $0 \leq t, s \leq 1$. If in addition, $F_2 \overset{C}{<} G$, then $\frac{H_{F_2}^{-1}(t,s)}{H_{F_2}^{-1}(1,1)} \geq ts$; $0 \leq t, s \leq 1$.*

Proof. If $F_1 \overset{C}{<} F_2$, then $\frac{f_1(x,y)}{f_2[f_2^{-1}F_1(x,y)]}$ is increasing in x and increasing in y. This implies that $\frac{f_1\left[\left(F_1^{-1}(u), F_1^{-1}(v)\right)\right]}{f_2\left[\left(F_2^{-1}(u), F_2^{-1}(v)\right)\right]}$ is increasing in u and increasing in v. Hence

$$
\begin{aligned}
\frac{H_{F_1}^{-1}(t,s)}{H_{F_1}^{-1}(1,1)} - \frac{H_{F_2}^{-1}(t,s)}{H_{F_2}^{-1}(1,1)} &= \int_0^t \int_0^s \left[\frac{1}{H_{F_1}^{-1}(1,1)} \times \frac{g\left(G^{-1}(u), G^{-1}(v)\right)}{f_1\left[\left(F_1^{-1}(u), F_1^{-1}(v)\right)\right]} - \frac{1}{H_{F_2}^{-1}(1,1)} \times \frac{g\left(G^{-1}(u), G^{-1}(v)\right)}{f_2\left[\left(F_2^{-1}(u), F_2^{-1}(v)\right)\right]} \right] dv\, du \\
&= \int_0^t \int_0^s \left[\frac{1}{H_{F_1}^{-1}(1,1)} \times \frac{f_2\left(F_2^{-1}(u), F_2^{-1}(v)\right)}{f_1\left[\left(F_1^{-1}(u), F_1^{-1}(v)\right)\right]} - \frac{1}{H_{F_2}^{-1}(1,1)} \right] \times \frac{g\left(G^{-1}(u), G^{-1}(v)\right)}{f_2\left[\left(F_2^{-1}(u), F_2^{-1}(v)\right)\right]} dv\, du \\
&= \int_0^t \int_0^s h(u,v) \times \frac{g\left(G^{-1}(u), G^{-1}(v)\right)}{f_2\left[\left(F_2^{-1}(u), F_2^{-1}(v)\right)\right]} dv\, du
\end{aligned}
$$

Since

$$
\int_0^1 \int_0^1 h(u,v) \times \frac{g\left(G^{-1}(u), G^{-1}(v)\right)}{f_2\left[\left(F_2^{-1}(u), F_2^{-1}(v)\right)\right]} dv\, du = 0
$$

and $h(u,v)$ has atmost one change of sign and if one change of sign actually occurs, it follows that,

$$
\int_0^t \int_0^s h(u,v) \times \frac{g\left(G^{-1}(u), G^{-1}(v)\right)}{f_2\left[\left(F_2^{-1}(u), F_2^{-1}(v)\right)\right]} dv\, du \geq 0
$$

and the first inequality follows. Since, $H_G^{-1}(t,s) = ts$, we have $\frac{H_{F_1}^{-1}(t,s)}{H_{F_1}^{-1}(1,1)} \geq \frac{H_{F_2}^{-1}(t,s)}{H_{F_2}^{-1}(1,1)}$; $0 \leq t, s \leq 1$. This completes the proof of the theorem. □

Remark 4.2. *$F \overset{*}{<} G$ is equivalent to F is BIFRA, if G is a bivariate exponential distribution.*

Theorem 4.3. *If $F \overset{*}{<} G$ and if G is a bivariate exponential distribution, then $\left(G^{-1}\circ F\right)(x,y)$ is Star-shaped.*

Proof. Let F be BIFRA, then $-\log \overline{F}(t,s)$ is star shaped. Also, $-\log\left[e^{-\lambda_1 t-\lambda_2 s-\lambda_{12}\max(t,s)}\right]$ is linear. Clearly, both the functions passing through the origin. Thus $-\log \overline{F}(t,s)$ crosses $-\log\left[e^{-\lambda_1 t-\lambda_2 s-\lambda_{12}\max(t,s)}\right]$ at most once and if a crossing occur, $-\log \overline{F}(t,s)$ crosses $-\log\left[e^{-\lambda_1 t-\lambda_2 s-\lambda_{12}\max(t,s)}\right]$ from below. It follows that $\overline{F}(t,s)$ crosses the bivariate exponential hazard function at most once and if it crosses, it does from above. It follows that $\left(G^{-1}\circ F\right)(x,y)$ is Star-shaped. This completes the proof of the theorem. □

Remark 4.4 (Geometric characterization of BNBU)**.** *The bivariate life distribution F with bivariate hazard function $R=\int_0^x\int_0^y r(u,v)dv\,du$ is BNBU if and only if R is super additive.*

Theorem 4.5. *The bivariate life distribution F is more BNBU than G if $\left(G^{-1}\circ F\right)(x,y)$ is super additive.*

Proof. The hazard function of $\left(G^{-1}\circ F\right)(x,y)$ is $R(x,y)=\frac{f(x,y)}{g\left[G^{-1}F(x,y)\right]}$. Consider

$$\begin{aligned}
R(x+t,y+s) &= \frac{f(x+t,y+s)}{g\left[G^{-1}F(x+t,y+s)\right]}\\
&= \frac{f\left[F^{-1}(u_1)+F^{-1}(u_2)\ ,\ F^{-1}(v_1)+F^{-1}(v_2)\right]}{g\left[G^{-1}(u_1)+G^{-1}(u_2)\ ,\ G^{-1}(v_1)+G^{-1}(v_2)\right]}\\
&\geq \frac{f\left[F^{-1}(u_1)\ ,\ F^{-1}(v_1)\right]+f\left[+F^{-1}(u_2)\ ,\ F^{-1}(v_2)\right]}{g\left[G^{-1}(u_1)+G^{-1}(u_1)\ ,\ G^{-1}(v_1)+G^{-1}(v_1)\right]}\\
&\geq \frac{f\left[F^{-1}(u_1)\ ,\ F^{-1}(v_1)\right]}{g\left[G^{-1}(u_1)+G^{-1}(u_1)\ ,\ G^{-1}(v_1)+G^{-1}(v_1)\right]}+\frac{f\left[+F^{-1}(u_2)\ ,\ F^{-1}(v_2)\right]}{g\left[G^{-1}(u_1)+G^{-1}(u_1)\ ,\ G^{-1}(v_1)+G^{-1}(v_1)\right]}\\
&\geq \frac{f\left[F^{-1}(u_1)\ ,F^{-1}(v_1)\right]}{g\left[G^{-1}F(x,y)\right]}+\frac{f\left[+F^{-1}(u_2)\ ,F^{-1}(v_2)\right]}{g\left[G^{-1}F(x,y)\right]}\\
&\geq \frac{f(x,y)}{g\left[G^{-1}F(x,y)\right]}+\frac{f(t,s)}{g\left[G^{-1}F(x,y)\right]}\\
&= R(x,y)+R(t,s).
\end{aligned}$$

Therefore, R is super additive and hence F is more BNBU than G. This completes the proof of the theorem. □

Theorem 4.6. *If G is a bivariate exponential distribution, then $F \overset{BNBUE}{<} G$ if and only if F is BNBUE.*

Proof. Let $\mathbf{X}=(X_1,X_2)$ be a random vector admitting absolutely continuous cdf in the support of the first quadrant $\{(x_1,x_2)/x_i\geq 0, i=1,2\}$ of the two-dimensional space $\mathbb{R}^2$ with $\mathbf{X}=(x_1,x_2)$ and $\mathbf{X}\geq\mathbf{x}$ denoting $X_i>x_i,\ i=1,2$ the bivariate mean residual life function of $\mathbf{X}$ is defined as $\mu_F(\mathbf{X})=E\left\{\mathbf{X}-\mathbf{t}\ /\ \mathbf{X}>\mathbf{t}\right\}$ where $\mathbf{t}=(t_1,t_2)$ is a vector of non-negative real numbers. The cdf of $\mathbf{X}$ is $\overline{F}(\mathbf{t})=Pr\ \{\mathbf{X}>\mathbf{t}\}=1-F_1(t_1)-F_2(t_2)+F(t_1,t_2)$ where, F_1 and F_2 are the cdf's of X_1 and X_2, respectively. Also

$$\begin{aligned}
\mathbf{F}(\mathbf{t}) &= \frac{F_2(t_2)\,\mu_{F_1}(0,t_2)}{\mu_{F_1}(\mathbf{t})}\ \exp\left\{-\int_0^{t_1}\frac{dx_1}{\mu_{F_1}(x_1,t_2)}\right\}\\
&= \frac{F_1(t_1)\,\mu_{F_2}(t_1,0)}{\mu_{F_2}(\mathbf{t})}\ \exp\left\{-\int_0^{t_2}\frac{dx_2}{\mu_{F_2}(t_1,x_2)}\right\},
\end{aligned}$$

where, $\mu_{F_i}(\mathbf{t})=E\left\{\mathbf{X}_i-\mathbf{t}_i\ /\ \mathbf{X}>\mathbf{t}\right\}\ :\ i=1,2$ is the ith component of $\mu_F(\mathbf{t})$. □

Theorem 4.7. *If G is a bivariate exponential distribution, then $F \overset{BDMRL}{<} G$ if and only if F is BDMRL.*

Proof. Let $\mathfrak{F}$ be the class of bivariate life distribution functions. Let $F, G \in \mathfrak{F}$ have bivariate mean residual life functions $\mu_F(x,y), \mu_G(x,y)$ and bivariate equilibrium survival functions

$$F_e(x,y) = \int_x^\infty \int_y^\infty \frac{\overline{F}(t,s)}{\mu_F(0,0)} ds\, dt \text{ and}$$
$$G_e(x,y) = \int_x^\infty \int_y^\infty \frac{\overline{G}(t,s)}{\mu_G(0,0)} ds\, dt$$

Define $W_F(u,v) = \overline{F} \circ F_e^{-1}(u,v)$ and $W_G(u,v) = \overline{G} \circ G_e^{-1}(u,v) \ : \ 0 \le u, v \le 1$. Here, W_F and W_G are proper bivariate life distribution functions. Since G is a bivariate exponential distribution, we have $W_G(u,v) = uv$. It follows that $\frac{\mu_F(F^{-1}(u,v))}{\mu_G(G^{-1}(u,v))}$ is equivalent to $W_F^{-1} \circ W_G(u,v)$ is star-shaped for $0 \le u, v \le 1$. This implies that $\frac{\mu_F(F^{-1}(u,v))}{\mu_G(G^{-1}(u,v))}$ is decreasing in $0 \le u, v \le 1$ and so F is BDMRL. This completes the proof of the theorem. □

Theorem 4.8. *If G is a bivariate exponential distribution, then $F \overset{BHNBUE}{<} G$ if and only if F is a BHNBUE distribution.*

Proof. Let G be a bivariate exponential distribution. Let $F \overset{BHNBUE}{<} G$, then

$$\frac{G_e^{-1} \circ F_e(x,y)}{xy} \ge \frac{\mu_G}{\mu_F} \ ; \quad \textit{for all } x, y \ge 0.$$

The rest of the proof is on similar lines of Theorem 4.4. □

Theorem 4.9. *If G is a bivariate exponential distribution, then $F \overset{BNBUFR}{<} G$ if and only if F is a BNBUFR distribution.*

Proof. F is a BNBUFR distribution, if $r(x,y) \ge r(0,0)$ for all $x, y > 0$.

$$\begin{aligned}
\left[\left(G^{-1} \circ F\right)(x,y)\right]' &\ge \left[\left(G^{-1} \circ F\right)(0,0)\right]' \\
&\ge \left[G^{-1}\left(F(0,0)\right)\right]' \\
&\ge \left[G^{-1}(0,0)\right]' \\
&\ge \left[(G(0,0))'\right]^{-1} \\
&= \left[(1)'\right]^{-1} \\
&= 0
\end{aligned}$$

This implies that, $\left(G^{-1} \circ F\right)$ is increasing in $x \ge 0$ and $y \ge 0$. It follows that, F is a BNBUFR distribution and the proof is complete. □

Theorem 4.10. *Let $Z_1, Z_2, \ldots, Z_n$, where $Z_i = (X_i, Y_i)$, be iid random variables with distribution $F(\cdot,\cdot)$ and F be BNBUA. Then $X_{(n)} = \max(X_1, X_2, \ldots, X_n)$ has the distribution $F_n(\cdot,\cdot)$ is BNBUA.*

Proof. F is BNBUA, if

$$\int_0^v \int_0^u \overline{F}(x+t, y+s)\, dt\, ds \le \overline{F}(x,y) \int_0^v \int_0^u \overline{F}(t,s)\, dt\, ds,$$

for $x, y, t, s \ge 0$ and u, v are finite. On substituting $x + t = \alpha$ and $y + s = \beta$, we have

$$\int_y^{v+y} \int_x^{u+x} \overline{F}(\alpha,\beta)\, d\alpha\, d\beta \le \overline{F}(x,y) \int_0^y \int_0^x \overline{F}(t,s)\, dt\, ds \ + \ \overline{F}(x,y) \int_y^{v+y} \int_x^{u+x} \overline{F}(\alpha,\beta)\, d\alpha\, d\beta$$

That is,

$$\int_y^{v+y} \int_x^{u+x} \frac{\overline{F}(\alpha,\beta)}{\overline{F}(x,y)}\, d\alpha\, d\beta \le \int_0^y \int_0^x \overline{F}(t,s)\, dt\, ds \ + \ \int_y^{v+y} \int_x^{u+x} \overline{F}(\alpha,\beta)\, d\alpha\, d\beta.$$

This gives,

$$\int_y^{v+y} \int_x^{u+x} \left[\frac{\overline{F}(\alpha,\beta)}{\overline{F}(x,y)} - \overline{F}(\alpha,\beta)\right] d\alpha\, d\beta \le \int_0^y \int_0^x \overline{F}(\alpha,\beta)\, d\alpha\, d\beta.$$

Since F is a bivariate NBUA distribution, we have

$$\int_0^y \int_0^x \overline{F}(\alpha, \beta)\ d\alpha\, d\beta \leq \int_0^y \int_0^x \overline{F}^n(\alpha, \beta)\ d\alpha\, d\beta \tag{6.1}$$

and

$$\int_y^{v+y} \int_x^{u+x} \frac{\overline{F}(\alpha, \beta)}{\overline{F}(x, y)} \cdot F(x, y)\ d\alpha\, d\beta \geq \int_y^{v+y} \int_x^{u+x} \frac{\overline{F}^n(\alpha, \beta)}{\overline{F}^n(x, y)} \cdot F^n(x, y)\ d\alpha\, d\beta \tag{6.2}$$

This implies that

$$\int_y^{v+y} \int_x^{u+x} \left[\frac{\overline{F}(\alpha, \beta)}{\overline{F}(x, y)} \cdot F(x, y) - \frac{\overline{F}^n(\alpha, \beta)}{\overline{F}^n(x, y)} \cdot F^n(x, y)\right] d\alpha\, d\beta \geq 0$$

It follows that

$$\begin{aligned}
&\int_y^{v+y} \int_x^{u+x} \frac{\overline{F}(\alpha, \beta)}{\overline{F}(x, y)} \cdot F(x, y) \left\{1 - \overline{F}^{n-1}(\alpha, \beta) \frac{\overline{F}^n(\alpha, \beta)}{\overline{F}(\alpha, \beta)} \cdot \frac{\overline{F}(x, y)}{\overline{F}^n(x, y)}\right\} d\alpha\, d\beta \\
&\qquad = \int_y^{v+y} \int_x^{u+x} \frac{\overline{F}(\alpha, \beta)}{\overline{F}(x, y)} \cdot F(x, y) \times \left\{1 - \overline{F}^{n-1}(\alpha, \beta) \frac{1 + F(\alpha, \beta) + \cdots + F^n(\alpha, \beta)}{1 + F(x, y) + \cdots + F^n(x, y)}\right\} d\alpha\, d\beta \\
&\qquad \geq \int_y^{v+y} \int_x^{u+x} \frac{\overline{F}(\alpha, \beta)}{\overline{F}(x, y)} \cdot F(x, y) \times \left\{1 - \overline{F}^{n-1}(\alpha, \beta) \frac{1 + F(x, y) + \cdots + F^n(x, y)}{1 + F(x, y) + \cdots + F^n(x, y)}\right\} d\alpha\, d\beta \\
&\qquad \geq 0.
\end{aligned}$$

Since $F(x, y) \leq F(\alpha, \beta)$ for $x \leq \alpha$ and $y \leq \beta$. Hence, we have from equations (6.1) and (6.2)

$$\int_y^{v+y} \int_x^{u+x} F^n(x, y) \frac{\overline{F}^n(\alpha, \beta)}{\overline{F}^n(x, y)}\, d\alpha\, d\beta \leq \int_0^y \int_0^x \overline{F}^n(\alpha, \beta)\ d\alpha\, d\beta$$

If and only if

$$\int_y^{v+y} \int_x^{u+x} F_n(x, y) \frac{\overline{F}_n(\alpha, \beta)}{\overline{F}_n(x, y)}\, d\alpha\, d\beta \leq \int_0^y \int_0^x \overline{F}_n(\alpha, \beta)\ d\alpha\, d\beta.$$

Equivalently, we can write

$$\int_0^y \int_0^x \overline{F}_n(u + \alpha, v + \beta)\ d\alpha\, d\beta \leq \overline{F}_n(u, v) \int_0^y \int_0^x \overline{F}_n(\alpha, \beta)\ d\alpha\, d\beta,$$

so that, F_n is a bivariate NBUA. Thais completes the proof. □

5. Conclusion

In this paper, we have introduced some new class of bivariate life distributions and established the closure properties under the Formation of Coherent systems. We also studied some properties on Partial Ordering and Preservation Properties for some Bivariate Life Distributions.

References

[1] A.Abouammoh and Khalique, *On new renewal better than used classes of life distributions*, Statistics & Probability Letters, 48(2000), 189-194.

[2] R.E.Barlow and F.Proschan, *Mathematical Theory of Reliability*, John Wiley, New York, (1965).

[3] R.E.Barlow and F.Proschan, *Statistical Theory of Reliability and Life Testing*, Holt, Rinehart and Winston, (1975).

[4] J.D.Esary and F.Proschan, *Coherent structures of nonidentical components*, Technometrics, 5(1963), 191-209.

[5] Harshinder Singh and Kanchan Jain, *Preservation of Some Partial Orderings under Poisson Shock Models*, Advances in Applied Probability, 21(3)(1989), 713-716.

[6] W.Y.Loh, *A new generalization of the class NBU distributions*, IEEE Trans. Reliability, R-33(1984), 419-422.

[7] S.P.Mukherjee and A.Chatterjee, *On Some Properties of the Multivariate Aging Classes*, Probability in the Engineering and Informational Sciences, 5(04)(1991), 523-534.

[8] Syed Tahir Hussainy, *On Some Aging Properties of Bivariate Life Distributions under Convolution*, Int. J. of Science and Humanities, 1(1)(2015), 47-52.

[9] U.Rizwan and Syed Tahir Hussainy, *A New Class of Life Distribution based on Characteristic function ordering*, Int. J. of Science and Humanities, 1(2)(2015), 146-153.

[10] U.Rizwan and Syed Tahir Hussainy, *Moment inequalities for some Bivariate Ageing classes*, Recent Trend in Mathematics, (2015), 235-240.

[11] Zohdy M. Nofal, *A New Bivariate Class of Life Distributions*, Applied Mathematical Sciences, 7(2)(2013), 49-60.