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Pseudo dachromatic Index of Graphs

Research Article

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Abstract: By extending the notion of pseudo d-achromatic number in the context of (k, d)-coloring and introduced the concept of pseudo d-achromatic number ψ_s^d(G) of a graph G. In this paper, I introduce the concept of pseudo d-achromatic index $\psi_s^{d'}$ (G) in the context of (k, d)-edge coloring of G and discuss the upper bound for this parameter and determine its characterization and provide several values of this pseudo d-achromatic index of some families of graphs.

Keywords: (k, d)-edge coloring, k-pseudo edge complete d-coloring, pseudo d-achromatic index, pseudo d-achromatic edge k coloring.

1. Introduction

Let G be a simple graph. A coloring of its vertices C : V(G) → {1,2,3,. . . ,k} is pseudo complete if every pair of different colors appears in an edge. The pseudo achromatic number ψ_s(G) is the maximum k for which there exists a pseudo complete coloring of G. If the coloring is required to be proper (that is each chromatic class is independent) then such a maximum is known as the achromatic number of G which is denoted by ψ(G). The chromatic number χ(G) is the minimum number of colors required for a proper vertex coloring of G. A chromatic coloring that used χ(G)-colors is a complete coloring. Hence $\chi(G) \leq \psi(G) \leq \psi_s(G)$ [1]. Vince [6] introduced the concept of star chromatic number which is the natural generalization of chromatic number. Let k and d be positive integers with $k \geq 2d$. Let Z_k = {1,2,. . . ,k} be the set of integers modulo k and D_k(x, y) = min{$|x-y|$, k - $|x-y|$}. Then a (k, d)-coloring of a graph G is a mapping C : V(G) → Z_k such that D_k(C(u),C(v)) ≥ d for each edge $uv \in E$. Similarly the concept of (k, d)-edge coloring of G is a mapping C : E(G) → Z_k such that D_k(C(e_i),C(e_j)) ≥ d for every adjacent edges e_i, $e_j \in E(G)$.

The concept of pseudo complete d-coloring and pseudo d-achromatic number of a graph G in the context of (k, d)-coloring was introduced in [5]. In this paper, I extend the analogus result in the context of (k, d)-edge coloring of G. Now I introduce the pseudo d-achromatic index of a graph G. and present some basic results on this parameter in the context of (k, d)-edge coloring of G.

2. Main Results

Definition 2.1. *Let k and d be two positive integers with $k \geq 2d$. A pseudo edge complete d-coloring of G using k colors is a mapping $\varphi : E(G) \to Z_k$ such that for any two colors $i, j \in Z_k$ with $D_k(i,j) \geq d$ there exists adjacent edges e_x, e_y*

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such that $\varphi(e_x) = i$ and $\varphi(e_y) = j$. A graph having a pseudo edge complete d-coloring using k colors is called k-pseudo edge complete d-colorable graph. The maximum value of k for which G is k-pseudo edge complete d-colorable is called the pseudo edge d-achromatic number or pseudo d-achromatic index of G and is denoted by $\psi_s^{d'}(G)$. If pseudo edge complete d-coloring of G proper, then such a maximum value of k is called d-achromatic index of G which is denoted by $\psi^{d'}(G)$.

Result 2.2. *The pseudo d-achromatic index of a graph G also interpreted as the pseudo d-achromatic number of the line graph L(G) of a graph G. That is $\psi_s^{d'}(G) = \psi_s^d[L(G)]$.*

Example 2.3. *The Petersen graph and three dimensional hyper cube are 7-pseudo edge 2-colorable graphs as shown in figure 1 and figure 2.*

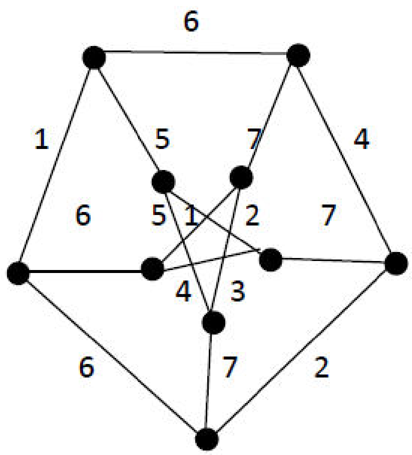

Figure 1. **The (10, 15)-Peterson graph**

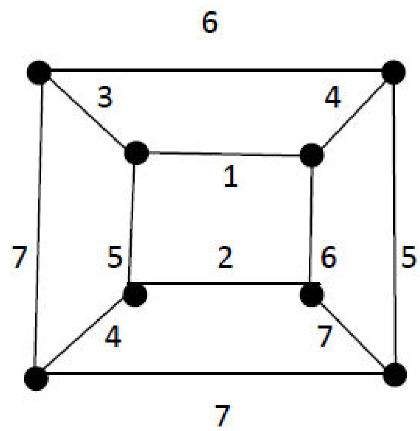

Figure 2. **Three dimensional hyper cube graph**

Result 2.4. *Let k and d be positive integers with $k \geq 2d$. Consider the graph $G_k^d = (V, E)$ where V= $\{v_1, v_2, \ldots, v_k\}$ and E=$\{(v_i, v_j) \,/\, D_k(i, j) \geq d\}$. Clearly G_k^d is a (k-2d+1)-regular graph and the size of the graph is $\frac{k(k-2d+1)}{2}$ and G_k^d is k-pseudo edge complete d-colorable graph. The following figure 3 shows G_6^2 is 6-pseudo edge 2-colorable therefore $\psi_s^{2'}(G_6^2) \geq 6$ where as $\psi^{2'}(G_6^2) = 5 = \chi^{2'}(G_6^2)$ as shown in figure 4.*

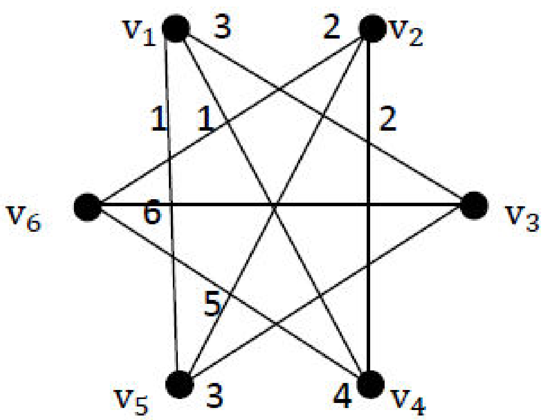

Figure 3.

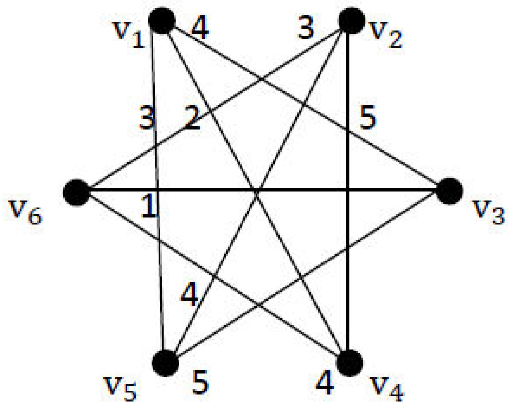

Figure 4.

Result 2.5. *If G admits k-pseudo edge complete d-coloring, then for any pair of adjacent edges, having colors i and j with* $D_k(i,\ j) \geq d$ *there exists atleast one vertex.*

Theorem 2.6. *The star graph* $K_{1,q}$ *(that is the complete bipartite graph with bipartition (X, Y) of the vertex set such that* $|X| = 1$ *and* $|Y| = q$*) is q-pseudo edge complete d-colorable graph with* $q \geq 2d$ *and* $\psi_s^{d'}(K_{1,q}) = q$*.*

Proof. The vertex set of the star graph has bipartition (X, Y) where X= {v} and Y = {v_1,v_2,...,v_q } and E($K_{1,q}$) = { vv_1,vv_2,...,vv_q} then f : E($K_{1,q}$) → {1,2,...,q} defined by f(vv_i) = i where $1 \leq i \leq q$ gives a q-pseudo edge complete d-coloring of $K_{1,q}$. Hence $\psi_s^{d'}(K_{1,q}) \geq q$. Now claim $\psi_s^{d'}(K_{1,q}) \leq q$. Suppose $\psi_s^{d'}(K_{1,q})$ = q + 1 under some optimal pseudo edge complete d-coloring f. Then f will assign distinct colors to the edges of $K_{1,q}$. If q + 1^{th} color is assigned to one of the q edges, one color say i where $1 \leq i \leq q$ is left out so that atleast one color pair i, j of adjacent edges with $D_k(i,$ j) ≥ d, does not appear,a contradiction. Therefore $\psi_s^{d'}(K_{1,q}) \leq q$. Hence we get $\psi_s^{d'}(K_{1,q})$ = q. □

Proposition 2.7. *Let G be a k-pseudo edge complete d-colorable graph. Then* $|E(G)| \geq k\lceil \frac{k-2d+1}{\Delta} \rceil$ *where* $k \geq 2d$*, and* Δ*, maximum degree of the graph G.*

Proof. Since G is k-pseudo edge complete d-colorable graph. Then the line graph L(G) is k-pseudo complete d-colorable graph. Consider any k-pseudo complete d-coloring of L(G), for any color c, there exists k-2d+1 edges in L(G) such that one end vertex of each of these k-2d+1 edges receive the color c. Hence there must be atleast $\lceil \frac{k-2d+1}{\Delta} \rceil$ vertices having the color c so that $|V(L(G))| \geq k\lceil \frac{k-2d+1}{\Delta} \rceil$. Therefore, $|E(G)| \geq k\lceil \frac{k-2d+1}{\Delta} \rceil$ since $V(L(G)) = E(G)$ for any graph G. Hence the result. □

Corollary 2.8. *For any graph with maximum degree* Δ, $\psi_s^{d'}(G) \leq \max\{k/k\lceil \frac{k-2d+1}{\Delta} \rceil \leq |E(G)|\}$.

Theorem 2.9. *Let k and d be positive integers with* $k \geq 2d$, *then* $\psi_s^{d'}(G_{2d+1}^d) = 2d + 1$.

Proof. Let the vertex set of V(G_{2d+1}^d) = { v_1,v_2,..., v_{2d+1} } and E(G_{2d+1}^d) = {(v_i, v_j)/$D_k(i$, j) ≥ d}. Clearly G_{2d+1}^d is a regular graph of degree 2 and the size of the graph is 2d+1. Hence G_{2d+1}^d is itself a cycle C = { v_{d+1}, v_1, v_{d+2}, v_2, v_{d+3},..., $v_{2d+1}v_{d+1}$} consisting of 2d +1 edges. Let the edge set of C be { $e_1 = v_{d+1}v_1$,$e_2 = v_1v_{d+2}$, $e_3 = v_{d+2}v_2$,$e_4 = v_2v_{d+3}$, ...,e_{2d+1} = $v_{2d+1}v_{d+1}$} and φ : E(G_{2d+1}^d) → {1,2,................., 2d + 1} defined by $\varphi(e_1)$ = 1 and $\varphi(e_i)$ = (i - 1) + d mod(2d + 1) where $2 \leq i \leq 2d+1$ gives a (2d +1)-pseudo edge complete d-coloring of G_{2d+1}^d. Hence

$$\psi_s^{d'}(G_{2d+1}^d) \geq 2d + 1. \quad (1)$$

By corollary 2.8, we have $\psi_s^{d'}(G_{2d+1}^d) \leq \max\{2d+1/(2d+1)\lceil \frac{2d+1-2d+1}{2} \rceil \leq |E(G_{2d+1}^d)|\}$. Hence

$$\psi_s^{d'}(G_{2d+1}^d) \leq 2d + 1. \quad (2)$$

Hence we have $\psi_s^{d'}(G_{2d+1}^d) = 2d+1$ by (1) and (2). □

Theorem 2.10. *Let k and d be positive integers such that* $k \geq 2d$. *Let n(k, d) denote the integer* $\frac{k(k-2d+1)}{2}$ *or* $\frac{k(k-2d+1)}{2} + \frac{k}{2}$ *according as k is odd or even. Then the cycle C on n(k,d) vertices is k-pseudo edge complete d-colorable graph.*

Proof.

Case (I): k is odd

Consider the graph G_k^d on k vertices {v_1, v_2, ..., v_k} which is (k-2d+1)-regular. Since k is odd, G_k^d is an Eulerian graph. Consider the euler tour T of G_k^d which will contain $\frac{k(k-2d+1)}{2}$ edges and has the same starting and end vertices. Let it be { x_1, x_2, ..., $x_{k(k-2d+1)}$}. Consider a cycle on $\frac{k(k-2d+1)}{2}$ vertices and hence $\frac{k(k-2d+1)}{2}$ edges. Let its edge set E = { e_1,e_2,...,e_k,...,$e_{\frac{k(k-2d+1)}{2}}$ }. For $1 \leq i \leq \frac{k(k-2d+1)}{2}$, $\varphi : E(C) \to \{1, 2, 3, \ldots, k\}$ defined by $\varphi(e_i) = x_i$ gives k-pseudo edge complete d-coloring of the cycle C on n(k, d)= $\frac{k(k-2d+1)}{2}$ vertices.

Case (II): k is even

If k = 2d, then n = k and the number of pairs $i, j \in \{1, \ldots, k\}$ with $D_k(i$, j) ≥ d is exactly $\frac{k}{2}$. Since the cycle C containing an independent set of $\frac{k}{2}$ vertices, it follows C admits a k-pseudo edge complete d-coloring. Suppose $k > 2d$, consider the graph G_k^d with vertex set {v_1,v_2,...,v_k } and adding $\frac{k}{2}$ new edges ({$v_iv_{\frac{k}{2}+i}$}/1 ≤ i ≤ $\frac{k}{2}$) to G_k^d and the resulting graph is an eulerian graph and has an euler tour T. Then the euler tour T has $\frac{k(k-2d+1)}{2} + \frac{k}{2}$ edges say T = {x_1,x_2,...,$x_{\frac{k(k-2d+1)}{2}+\frac{k}{2}}$ }. Consider the cycle on $\frac{k(k-2d+1)}{2} + \frac{k}{2}$ vertices and hence $\frac{k(k-2d+1)}{2} + \frac{k}{2}$ edges. Let its edge set be E = {e_1,e_2,...,e_k,...,$e_{\frac{k(k-2d+1)}{2}}$,.........,$e_{\frac{k(k-2d+1)}{2}+\frac{k}{2}}$}.

For $1 \leq i \leq \frac{k(k-2d+1)}{2} + \frac{k}{2}$, $\varphi : E(C) \to \{1, \ldots, k\}$ defined by $\varphi(e_i) = x_i$ gives k-pseudo edge complete d-coloring of the cycle C on n(k, d) = $\frac{k(k-2d+1)}{2} + \frac{k}{2}$ vertices. Thus in all cases, we get a k-pseudo edge complete d-coloring of the cycle C on n(k, d) vertices where n(k, d) =$\frac{k(k-2d+1)}{2}$ if k is odd $\frac{k(k-2d+1)}{2} + \frac{k}{2}$ if k is even. □

Corollary 2.11. *Let k and d be positive integers with k ? 2d. Let n(k, d)* = $\frac{k(k-2d+1)}{2}$ *or* $\frac{k(k-2d+1)}{2} + \frac{k}{2}$ *according as k is odd or even. Then for the cycle C on n(k, d) vertices,* $\psi_s^{d'}(C_n) = \max\{k/n(k,d) \leq n\}$.

Example 2.12.

(1) *Consider the graph G_7^2 in figure 5. An euler tour of G_7^2 is given by (1,3,7,5,2,4,7,2,6,3,5,1,6,4,1). Since the line graph of T ((ie) L(T) is T itself and so E(T) = V(L(T)) = V(T). Hence the color i^{th} vertex of T is replaced by the color of i^{th} edge of T. The 7-pseudo edge complete 2-coloring of the cycle corresponding to the above euler tour is given in figure 6.*

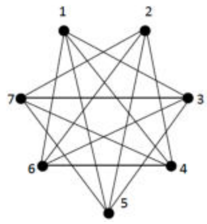

Figure 5.

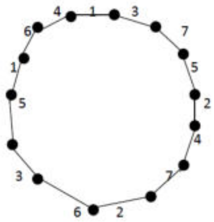

Figure 6.

(2) *Consider the multigraph G given in figure 7 which is obtained from G_6^2 by adding new edges $\{1, 4\}$, $\{2, 5\}$, $\{3, 6\}$. An euler tour T of G is given by (1,4,1,5,2,4,6,3,5,2,6,3,1) since the line graph L(T) is T itself the color i^{th} vertex of T is replaced by the i^{th} edge of T. The 6-pseudo edge complete 2-colorability of C_{12} corresponding to the above euler tour given by figure 8.*

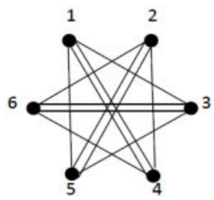

Figure 7.

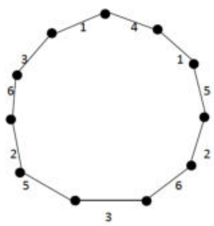

Figure 8.

Proposition 2.13. *Let k and d be positive integers with $k \geq 2d$. Let*

$$n(k,d) = \begin{cases} \frac{k(k-2d+1)}{2}, & \text{if } k \text{ is odd;} \\ \frac{k(k-2d+1)}{2} + \frac{k}{2} - 1, & \text{if } k \text{ is even.} \end{cases}$$

Then any path P on n(k, d) + 1 vertices is k-pseudo edge complete d-colorable.

Proof.

Case (I): k is odd

From an eulerian graph G_k^d, consider an euler tour containing $\frac{k(k-2d+1)}{2}$ edges say $\{x_1, x_2, \ldots, x_{\frac{k(k-2d+1)}{2}}\}$. Consider a cycle C=$\{v_1, v_2, \ldots, v_{\frac{k(k-2d+1)}{2}}, v_1\}$ on $\frac{k(k-2d+1)}{2}$ edges. From the cycle C, the path P can be constructed with $\frac{k(k-2d+1)}{2} + 1$ vertices where $v_{\frac{k(k-2d+1)}{2}+1} = v_1$ and hence $\frac{k(k-2d+1)}{2}$ edges let it be $\{e_1, e_2, \ldots, e_{\frac{k(k-2d+1)}{2}}\}$.
For $1 \leq i \leq \frac{k(k-2d+1)}{2}$, let $\varphi : E(P) \to \{1, 2, 3, \ldots, k\}$ defined by $\varphi(e_i) = x_i$ gives k-pseudo edge complete d-coloring on the path P on $n(k,d) + 1 = \frac{k(k-2d+1)}{2} + 1$ vertices.

Case (II): k is even

With G_k^d, add $\frac{k}{2}$ new edges $\left(\{v_i v_{\frac{k}{2}+i}\}/1 \leq i \leq \frac{k}{2}\right)$ so that the resulting graph G is eulerian in which the euler tour T = $\{x_1, x_2, \ldots, x_{\frac{k(k-2d+1)}{2}+\frac{k}{2}}\}$ containing $\frac{k(k-2d+1)}{2} + \frac{k}{2}$ edges.
From the cycle $C_n = \{v_1, v_2, \ldots, v_{\frac{k(k-2d+1)}{2}+\frac{k}{2}}, v_1\}$ having $\frac{k(k-2d+1)}{2} + \frac{k}{2}$ edges say $\{e_1, e_2, \ldots, e_{\frac{k(k-2d+1)}{2}+\frac{k}{2}}\}$ a path P_n on $\frac{k(k-2d+1)}{2} + \frac{k}{2}$ can be constructed by deleting any new edge which is added (ie) $P_n = C_n - e$.
For $1 \leq i \leq \frac{k(k-2d+1)}{2} + \frac{k}{2}$, let $\varphi : E(P_n) \to \{1, 2, 3, \ldots, k\}$ defined by $\varphi(e_i) = x_i$ gives k-pseudo edge complete d-coloring on the path P_n on $n(k,d) + 1 = \frac{k(k-2d+1)}{2} + \frac{k}{2}$ vertices. □

Lemma 2.14. *If $\psi_s^{d'}(C_p) = n$ and $k \geq 2$ then $\psi_s^{d'}(C_{p+k}) \geq n$.*

Proof. Consider the edges of C_{p+k} labelled as $e_1, e_2, \ldots, e_p, \ldots, e_{p+k}$ assigns to each edge $e_1, e_2, \ldots, e_p$ the color it receives in a pseudo edge complete d-coloring of C_p using n colors. It is then suffices to assign colors from 1,…,n to the edges $e_{p+1}, \ldots, e_{p+k}$ so that the result is a valid pseudo edge complete d-coloring. Suppose e_1 assigned color α and e_p assigned color β. If k is even, assign α to $e_{p+1}, e_{p+3}, \ldots, e_{p+k-1}$ and β to $e_{p+2}, e_{p+4} \ldots, e_{p+k}$. If k is odd, assign α to $e_{p+1}, e_{p+3}, \ldots, e_{p+k-2}$ and β to $e_{p+2}, e_{p+4} \ldots, e_{p+k-1}$ and any other color to e_{p+k}. Thus C_{p+k} is n-pseudo edge complete d-colorable. Hence $\psi_s^{d'}(C_{p+k}) \geq n$. □

Proposition 2.15. *Let k and d be positive integers with $k \geq 2d$. If $n = \frac{k(k-2d+1)}{2} + \frac{k}{2}$ and k is even. Then $\psi_s^{d'}(C_n) = \psi_s^{d'}(C_{n+1})$ =k.*

Proof. By proposition 2.10 and Corollary 2.11, we have proved already, $\psi_s^{d'}(C_n)$ = k if n = $\frac{k(k-2d+1)}{2} + \frac{k}{2}$ and k is even. Now to prove $\psi_s^{d'}(C_n) = \psi_s^{d'}(C_{n+1})$ = k. In the pseudo edge complete d-coloring of the cycle C_n with n = $\frac{k(k-2d+1)}{2} + \frac{k}{2}$ vertices using k colors where k is even, the adjacency between the colors assigned to the edges e_i and e_j is repeated elsewhere in the cycle. Now remove the vertex in which the edges e_i and e_j are incident, and replace it by an edge, colored within k colors differently from e_i and e_j, we get a k-pseudo edge complete d-coloring of the cycle C_{n+1} so that $\psi_s^{d'}(C_n) = \psi_s^{d'}(C_{n+1})$ = k. □

Proposition 2.16. *For $k \geq 2d$ and k is odd, let $n = \frac{k(k-2d+1)}{2}$. Then $\psi_s^{d'}(C_n) = k$ but $\psi_s^{d'}(C_{n-1}) = k - 1$.*

Proof. By proposition 2.10 and Corollary 2.11, we have proved $\psi_s^{d'}(C_n) = k$ if $n = \frac{k(k-2d+1)}{2}$ and k is odd. Suppose that for n = $\frac{k(k-2d+1)}{2}$, $\psi_s^{d'}(C_{n+1}) = k$. Since $n + 1 = \frac{k(k-2d+1)}{2} + 1, n - 1$ colors appears $\frac{k-2d+1}{2}$ times, and one color say α appears $1 + \frac{k-2d+1}{2}$ times. Then α must repeat an adjacency to some other color, but there is no color available which could be adjacency to α. Thus $\psi_s^{d'}(C_n) = k - 1$. Hence the result. □

Definition 2.17. *If $\psi_s^{d'}(G) = k$ and $\psi_s^{d'}(G - e) < k$ for every edge $e \in E(G)$ then G is called k-edge d-minimal.*

Example 2.18.

1. *Any cycle C_n where $n = \frac{k(k-2d+1)}{2}$ where $k \geq 2d$ and k is odd is k-edge d-minimal whereas any cycle C_n where $n = \frac{k(k-2d+1)}{2} + \frac{k}{2}$ and k is even is not k-edge d-minimal since $\psi_s^{d'}(C_n) = \psi_s^{d'}(C_n - e) = k$.*

2. *Any star graph $K_{1,q}$ where $q \geq 2d$ is q-edge d-minimal since $\chi_s^{d'}(K_{1,q}) = q$ and $\chi_s^{d'}(K_{1,q} - e) = q - 1$.*

Lemma 2.19. *For every k, there is a graph G and an independent set S of points of G such that $\psi_s^{d'}(G) - \psi_s^{d'}(G - S) \geq k$.*

Proof. Let $n \geq k$ be even. Then if $p = \frac{n(n-2d+1)}{2} + \frac{n}{2}$ is even then the cycle C_p is bipartite by the theorem, " A graph is bipartite if and only if all of its cycles are of even length". The points u_1,u_3,...,u_{p-1} form an independent set of S such that $C_p - S$ is totally disconnected. Therefore the induced subgraph $<C_p - S>$ is an empty graph. Hence $\psi_s^{d'}(C_p) - \psi_s^{d'} < C_p - S >= \psi_s^{d'}(C_p) = n$ by proposition 2.10. Therefore $\psi_s^{d'}(C_p) - \psi_s^{d'}(C_p - S) = n \geq k$. Hence the result. Let P_p be the path with p points. □

Lemma 2.20. *If $r > s$, then $\psi_s^{d'}(P_r) \geq \psi_s^{d'}(P_s)$.*

Proof. It is sufficient to prove $\psi_s^{d'}(P_{r+1}) \geq \psi_s^{d'}(P_r)$. Let u_1 be the end point of P_{r+1} and let u_3 be at distance 2 from u_1, then identifying u_1 with u_3 defines a homomorphism from P_{r+1} to P_r. If $\psi_s^{d'}(P_r)$ = n then P_{r+1} defines a n-pseudo edge complete d-coloring so that $\psi_s^{d'}(P_{r+1}) \geq n = \psi_s^{d'}(P_r)$. □

Proposition 2.21. *For $n \geq 2d$, let $p = \frac{n(n-2d+1)}{2} + \frac{n}{2}$ if n is even, $\psi_s^{d'}(P_{p-1}) < \psi_s^{d'}(P_p) = n$. If $p = \frac{n(n-2d+1)}{2}$ and n is odd, $\psi_s^{d'}(P_p) < \psi_s^{d'}(P_{p+1}) = n$.*

Proof. We first examine the equalities, if n is even then $\psi_s^{d'}(C_p)$ = n by proposition 2.10 and some adjacency is repeated. Thus $P_p = C_p$ − x where x is any new edge that is added in defining n-pseudo complete d-coloring of C_p then $\psi_s^{d'}(P_p)$ = n. If n is odd, we obtain a n-pseudo edge complete d-coloring of P_{p+1} from one for C_p by removing some point u, adjacent to the points v_1 and v_2 and then adding new points u_1, u_2 and the lines $u_1v_1$and u_2v_2.
By assigning to the lines u_1v_1, u_2v_2 the color assigned to the lines uv_1 and uv_2, we obtain a n-pseudo edge complete d-coloring for P_{p+1}. For the inequality, in the even case, we note that the n pseudo edge complete d-coloring of P_{p-1} give rise to one for C_{p-1} if the end lines of the path colored differently, and one for C_{p-2} if they are colored the same. Each of these would violate the result of proposition 2.15 when n is odd, P_p has only $\frac{n(n-2d+1)}{2} - 1$ lines, whereas atleast $\frac{n(n-2d+1)}{2}$ lines would be required if the graph were to have a n-pseudo edge complete d-coloring. □

3. Upper Bounds for Pseudo d-achromatic Index of Graphs

Hung-Lin Fu in his paper [2] gave another approach to the pseudo achromatic index and achromatic index of a graph G and studied upper bounds for them. Now I define pseudo d-achromatic index $\psi_s^{d'}$(G), d-achromatic index $\psi^{d'}$(G) in terms of decomposition graphs.
Let G be a (V, E) graph. A collection D = {E_1,E_2,...,E_n } of non empty subsets of E is a decomposition of G, if E is the disjoint union of E_1,E_2,...,E_n.
If every set in the decomposition D of G is a matching, we say D is a proper decomposition of G. The decomposition graph D(G) is defined as follows.

1. $V(D(G)) = D$ and

2. For i $\neq$ j, $\{E_i, E_j\}$? E(D(G)) if and only if $V(E_i)_G \cap V(E_j)_G \neq \varphi$ where $(E_i)_G$ is the edge induced subgraph of G.

If D(G) $= G_k^d$ where $k \geq 2d$, then we say the decomposition of G is complete with respect to the (k, d)-edge coloring of G. A pseudo d-achromatic edge k coloring of G is a complete decomposition of G into $D = \{E_1, E_2, \ldots, E_k\}$ with respect to (k, d)-edge coloring of G. A proper pseudo 2-achromatic edge 7-coloring of G is shown in figure 6 which is a proper complete decomposition of G into D = { $E_1 = \{e_1, e_{12}\}$, $E_2 = \{e_5, e_8\}$, $E_3 = \{e_2, e_{10}\}$, $E_4 = \{e_6, e_{14}\}$, $E_5 = \{e_4, e_{11}\}$, $E_6 = \{e_9, e_{13}\}$, $E_7 = \{e_3, e_7\}$ } having colors 1,2,3,4,5,6,7 respectively with respect to (7, 2)-edge coloring of G.

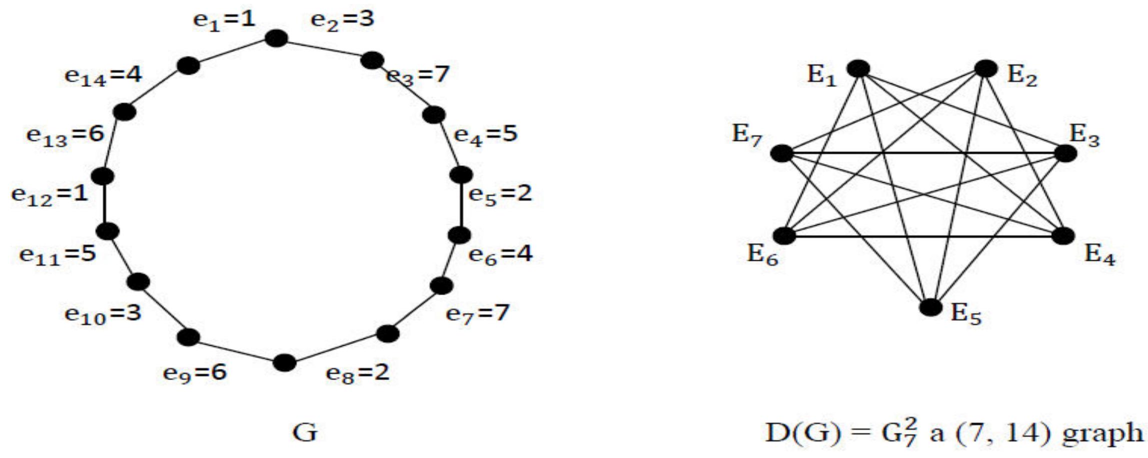

Figure 9.

Definition 3.1. *The pseudo d-achromatic index $\psi_s^{d'}(G)$ of a graph G is largest k such that G has a pseudo d-achromatic edge k-coloring of G. Now $\psi^{d'}(G)$ is the largest m such that G has a proper complete decomposition of G with respect to the (m, d)-edge coloring of G where as $\chi^{d'}(G)$ is the smallest m such that G has a proper complete decomposition of G with respect to the (m, d)-edge coloring of G. The above figure 9 shows that $\chi^{2'}(G) = \psi^{2'}(G) = 7$.*

The following lemmas are easy to check.

Lemma 3.2. *Let G be a graph. Then $\chi^{d'}(G) \leq \psi^{d'}(G) \leq \psi_s^{d'}(G)$.*

Lemma 3.3. *Let H be a subgraph of G. Then $\psi_s^{d'}(H) \leq \psi_s^{d'}(G)$.*

Theorem 3.4. *For any graph G, $\psi^{d'}(G) \leq \psi_s^{d'}(G) \leq \lfloor \frac{e(G)+(2d-1)\chi^{d'}(G)}{2d} \rfloor$.*

Proof. By lemma 3.3, it suffices to prove $\psi_s^{d'}(G) \leq \lfloor \frac{e(G)+(2d-1)\chi^{d'}(G)}{2d} \rfloor$. Let $\chi^{d'}(G) = n$. Then there exists a proper complete decomposition D = { $E_1, E_2, \ldots\ldots\ldots\ldots\ldots\ldots, E_n$ } where $n \geq 2d$, with respect to (n, d)-edge coloring of G. Let $\psi_s^{d'}$(G) = m and $D' = \{E_1, E_2, \ldots, E_m$ } where $m \geq 2d$ be a complete decomposition of G with respect to (m, d)-edge coloring of G. It is clear that for $1 \leq i \neq j \leq m$, $F_i \cup F_j \not\subset E_k$ where $k = 1, 2, 3, \ldots, n$.
Hence atleast m-n sets of D' contains atleast 2d edges from different sets of D. Then, $2d(m-n) + n \leq e(G)$. Therefore, $2dm - 2dn + n \leq e(G)$ where e(G) is the number of lines of G. Hence, $2dm \leq e(G) + (2d-1)n$ which implies, $m \leq \lfloor \frac{e(G)+(2d-1)n}{2d} \rfloor$ so that $\psi^{d'}(G) \leq \psi_s^{d'}(G) \leq \lfloor \frac{e(G)+(2d-1)\chi^{d'}(G)}{2d} \rfloor$. □

Example 3.5. *Consider a three dimensional hyper cube G, second image of figure 10 shows that $\chi^{2'}(G) = \psi^{2'}(G) = 6$ where as $\psi_s^{2'}(G) = 7$ as shown in second image of figure 10.*

In second image of figure 10, the complete decomposition of G in D = { $E_1 = \{e_1\}$, $E_2 = \{e_2\}$, $E_3 = \{e_9\}$, $E_4 = \{e_9, e_{12}\}$, $E_5 = \{e_4, e_6\}$, $E_6 = \{e_3, e_5\}$, $E_7 = \{e_7, e_{11}, e_8\}$ } *having colors 1, 2, 3, 4, 5, 6, 7 respectively. Here also* $D(G) = G_7^2$ *so that*

$$\psi_s^{2'}(G) \geq 7. \tag{3}$$

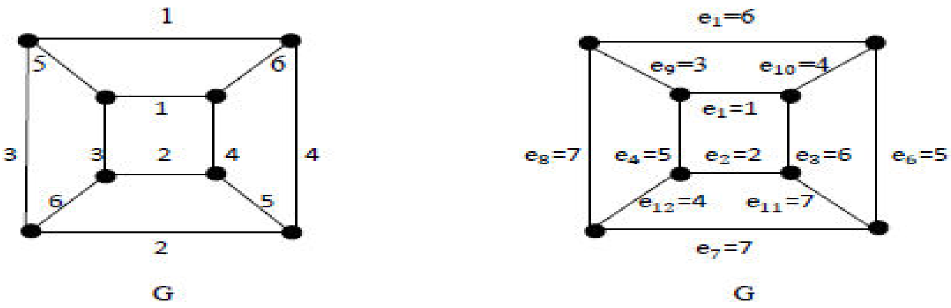

Figure 10.

From Theorem 3.4, second image of figure 10 shows that

$$\psi_s^{2'}(G) \leq \lfloor \frac{12+36}{4} \rfloor \leq \lfloor \frac{30}{4} \rfloor = 7. \tag{4}$$

(3) *and* (4) *implies that* $\psi_s^{2'}(G) = 7$.

Remark 3.6.

(A) *By considering the star graph* $K_{1,q}$ *(that is the complete bipartite graph) with bipartition graph (X, Y) of the vertex set such that* $|X| = 1$ *and* $|Y| = q$.

It has been proved $\chi^{d'}(K_{1,q}) = q$ *and* $\psi_s^{d'}(K_{1,q}) = \lfloor \frac{q+3q}{4} \rfloor = q$ *for every positive integer* $q \geq 2d$ *(from Theorem 3.4). This shows that the upper bound in the Theorem 3.4 is best possible.*

(B) *Even though the upper bound obtained in Theorem 3.4 is the best possible, the difference between* $\psi_s^{d'}(G)$ *and* $\lfloor \frac{e(G)+(2d-1)\psi^{d'}(G)}{2d} \rfloor$ *can be bigger than any positive integer. For example, consider the double star* $S_{p,q}$ *where* $p \geq q \geq 2d$ *as described in figure 11 (that is* $K_{1,p}$ *and* $K_{1,q}$ *which share an edge). Then* $\psi^{d'}(S_{p,q}) = p$, $e(G) = p + q - 1$ *and* $\psi_s^{d'}(S_{p,q}) = p$. *Hence the difference between* $\psi_s^{d'}(G)$ *and* $\lfloor \frac{e(G)+(2d-1)\psi^{d'}(G)}{2d} \rfloor$ *is* $\lfloor \frac{q-1}{2d} \rfloor$ *and can be made arbitrarily large by suitable choices of p, q and d.*

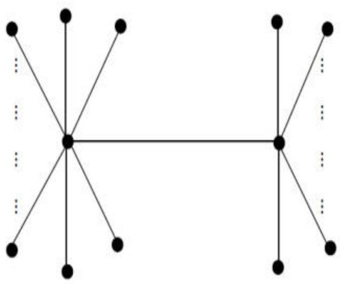

Figure 11.

Another upper bound can be obtained in a different way.

Theorem 3.7. *Let G be a graph of order p and size e, with maximum degree* Δ*,* $\Delta \geq 2d$*. Then,* $\psi_s^{d'}(G) \leq \max_{1 \leq k \leq \lfloor \frac{p}{2d} \rfloor} \left(\min \lfloor \frac{p\Delta}{2k} \rfloor, 2k(\Delta - 1) + 2d - 1\right)$.

Proof. Let f be a pseudo d-complete edge $\psi_s^{d'}(G)$-coloring of G. By Vizings Theorem, and by Lemma 3.3 we get $\psi_s^{d'}(G) \geq \Delta d$. Suppose that the smallest color class S of f consists of k edges. Then $1 \leq k \leq \lfloor \frac{e}{\Delta d} \rfloor$. Since the edge induced subgraph $< S >_G$ has atmost 2k vertices and the degree of each vertex is atmost Δ. The number of edges not in S but incident with some edges in S is atmost $2k(\Delta - 1)$.

Hence, $\psi_s^{d'}(G) \leq 2k(\Delta - 1) + 2d - 1$. On the other hand, since each color class consists of atleast k edges, the edge set E(G) can be decomposed into atmost $\lfloor \frac{p\Delta}{2k} \rfloor$ color classes. Therefore $\psi_s^{d'}(G) \leq \lfloor \frac{p\Delta}{2k} \rfloor$. Hence $\psi_s^{d'}(G) \leq \min\{\lfloor \frac{p\Delta}{2k} \rfloor, 2k(\Delta - 1) + 2d - 1\}$ and $\psi_s^{d'}(G) \leq \max_{1 \leq k \leq \lfloor \frac{e}{\Delta d} \rfloor} \left(\min\{\lfloor \frac{p\Delta}{2k} \rfloor, 2k(\Delta - 1) + 2d - 1\}\right)$. Since $\lfloor \frac{p\Delta}{2k} \rfloor$ is non increasing as a function of k and $\lfloor \frac{p\Delta}{2k} \rfloor \leq 2k(\Delta - 1) + 2d - 1$ if $k \geq \lfloor \frac{p}{2d} \rfloor$. Hence we have, $\psi_s^{d'}(G) \leq \max_{1 \leq k \leq \lfloor \frac{p}{2d} \rfloor} \left(\min\{\lfloor \frac{p?}{2k} \rfloor, 2k(\Delta - 1) + 2d - 1\}\right)$.

To see the upper bound in the above theorem is best possible, let us consider the graphs P_k and C_k the path and the cycle of order k respectively. We note that $\psi^{d'}(G) \leq \psi_s^{d'}(G) \leq m(G)$ where $m(G) = \max\{n/n.\lceil \frac{n-2d+1}{2(\Delta(G)-1)} \rceil \leq e(G)\}$ and it easy to check that m(G) is always larger than the upper bound in Theorem 3.7. The upper bound is appropriate in this case. The following result is known from Proposition 2.15, 2.16 and 2.21. □

Proposition 3.8. *Let* $m = \max\{n/n.\lceil \frac{n-2d+1}{2} \rceil \leq k\}$*. Then*

1. *For* $k \geq 2d$*,*

$$\psi_s^{d'}(P_{k+1}) = \begin{cases} m - 1, & \text{if } m \text{ is odd and } k = m.\lceil \frac{m-2d+1}{2} \rceil; \\ m, & \text{otherwise.} \end{cases}$$

2. *For* $k \geq 2d$*,*

$$\psi_s^{d'}(C_k) = \begin{cases} m - 1, & \text{if } m \text{ is odd and } k = m.\lceil \frac{m-2d+1}{2} \rceil + 1; \\ m, & \text{otherwise.} \end{cases}$$

Corollary 3.9. *For every* $k \geq 2d$*,* $\psi_s^{d'}(C_k) = m$ *where* $m = \max\{n/n.\lceil \frac{n-2d+1}{2} \rceil \leq k\}$*.*

Proof. We need only to show there is a pseudo d-achromatic edge m coloring of C_k for the case when m is odd and $k = m.\lceil \frac{m-2d+1}{2} \rceil + 1$. In that case, there is pseudo d-achromatic edge m coloring for the path P_{k+1} by identifying the first and last vertices, we get pseudo d-achromatic edge m-coloring of the cycle C_k. Hence the result. Obviously, by proposition 3.8 and corollary 3.9, the upper bound obtained in theorem 3.7 is the best possible. □

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