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Lie group analysis of two-dimensional variable-coefficient Potential Burgers equation

Research Article

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Abstract: The modern group analysis of differential equations is used to study a class of two-dimensional variable coefficient Potential Burgers equations. The group classification of this class is performed. We determine the one- and two-dimensional subalgebras of the symmetry algebra which is infinite-dimensional into conjugacy classes under the adjoint action of the symmetry group. Classification of its symmetry algebra into one- and two-dimensional sub-algebras are carried out in order to facilitate its reduction systematically to (1+1)-dimensional PDEs and then to first or second order ODEs.

Keywords: A variable coefficient Potential Burgers Equation, Symmetry algebra, Conjugacy class.

1. Introduction

The Burgers equation[6,7]

$$u_t = 2uu_x + u_{xx}, \tag{1}$$

is linearised to the heat equation

$$\phi_t = \frac{\delta}{2}\phi_{xx}, \tag{2}$$

through the Cole-Hopf transformation (Hopf [1] and Cole [2])

$$u = -\delta\frac{\phi_x}{\phi}. \tag{3}$$

Nimmo and Crighton [3] have derived Bäcklund transformations for nonlinear parabolic equations of the form

$$u_t - u_{xx} + H(t,x,u,u_x) = 0, \tag{4}$$

and proved that besides the Burgers equation (1) its inhomogeneous version, viz.,

$$u_t + uu_x + a(t,x) = \frac{\delta}{2}u_{xx}, \tag{5}$$

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is also linearisable. But Lighthill [4] showed that, unlike in (1) and (5), the viscosity is function of t.

In this paper, we consider the two-dimensional variable coefficient potential Burgers equation

$$u_t + uu_x + u_x^2 - A(t)u_{xx} - B(t)u_{yy} = 0, \tag{6}$$

where $A(t)$ and $B(t)$ are the arbitrary functions. Through the group classification of (6), we determine the values of $A(t)$ and $B(t)$. Hence we determined the following two equations,

$$u_t + uu_x + u_x^2 - Au_{xx} - Bu_{yy} = 0, \tag{7}$$

where A and B are constants.

$$u_t + uu_x + u_x^2 - Au_{xx} - e^{\alpha t}\sigma u_{yy} = 0, \tag{8}$$

where $B(t) = e^{\alpha t}\sigma$.

Thus, we investigate the symmetries and reductions of (7) and (8). We shall show that the above equation admits a symmetry groups and determine the corresponding Lie algebras, then classifies the one- and two- dimensional subalgebras of the symmetry algebras of (7) and (8) in order to reduce (7) and (8) to (1+1)-dimensional partial differential equations and then to ordinary differential equations. We shall establish that the symmetry generators form a closed Lie algebras and this allowed us to use the recent method due to Ahmad, Bokhari, Kara and Zaman [5] to successively reduce (7) and (8) to (1+1)-dimensional PDEs and ODEs with the help of two dimensional Abelian and solvable non-Abelian subalgebras.

2. The Symmetry Group & Lie Algebra of $u_t+uu_x+u_x^2-Au_{xx}-Bu_{yy} = 0$

If (7) is invariant under a one parameter Lie group of point transformations (Bluman and Kumei [2], Olver [5])

$$x^* = x + \epsilon\xi_1(x, y, t, u) + O(\epsilon^2), \tag{9}$$

$$y^* = y + \epsilon\xi_2(x, y, t, u) + O(\epsilon^2), \tag{10}$$

$$t^* = t + \epsilon\xi_3(x, y, t, u) + O(\epsilon^2), \tag{11}$$

$$u^* = u + \epsilon\phi_1(x, y, t, u) + O(\epsilon^2), \tag{12}$$

with the infinitesimal generator, $\xi_1 = \xi; \xi_2 = \eta, \xi_3 = \tau$ and $\phi_1 = \phi$, then the fourth prolongation $pr^{(4)}V$ of the corresponding vector field,

$$V = \xi(x, y, t; u)\partial_x + \eta(x, y, t; u)\partial_y + \tau(x, y, t; u)\partial_t + \phi(x, y, t; u)\partial_u, \tag{13}$$

satisfies

$$pr^{(4)}V\Omega(x, y, t; u)|_\Omega(x, y, t; u) = 0. \tag{14}$$

If $A(t)$ and $B(t)$ are constants, then the infinitesimal generator of V becomes

$$\begin{aligned} \xi &= c_2t + c_1, \\ \eta &= c_3, \\ \tau &= c_4, \\ \text{and} \quad \phi &= c_2, \end{aligned} \tag{15}$$

where c_1, c_2, c_3 and c_4 are constants. Thus, the Lie algebra of infinitesimal symmetries of the Potential Burgers Equation is spanned by the four vector fields

$$\begin{aligned} V_1 &= \partial_x, \\ V_2 &= t\partial_x + \partial_u, \\ V_3 &= \partial_y, \\ V_4 &= \partial_t. \end{aligned} \tag{16}$$

It is easy to check that the symmetry generators found in (7) form a closed Lie algebra whose commutation relations are given in Table 1.

The one-parameter groups $g_i(\epsilon)$ generated by the V_i where $i = 1, 2, 3, 4$ are

$$\begin{aligned} g_1(\epsilon) : (x, y, t; u) &\rightarrow (x + \epsilon, y, t, u)\ , \\ g_2(\epsilon) : (x, y, t; u) &\rightarrow (x + t\epsilon, y, t, u + \epsilon)\ , \\ g_3(\epsilon) : (x, y, t; u) &\rightarrow (x, y + \epsilon, t, u)\ , \\ g_4(\epsilon) : (x, y, t; u) &\rightarrow (x, y, t + \epsilon, u)\ , \end{aligned}$$

where $exp(\epsilon V_i)\ (x, y, t; u) = (\bar{x}, \bar{y}, \bar{t}; \bar{u})$ and

(i) g_1, g_2 and g_3 are the space-invariant,

(ii) g_4 is time translation of the equation.

Table 1. **Commutator Table**

$[V_i, V_j]$	V_1	V_2	V_3	V_4
V_1	0	0	0	0
V_2	0	0	0	$-V_1$
V_3	0	0	0	0
V_4	0	V_1	0	0

The commutation relations of the Lie algebra L, determined by V_1, V_2, V_3 and V_4 are shown in the above table.

$$[V_2, V_4] = -V_1 \text{ and } [V_4, V_2] = V_1$$

For this two-dimensional Lie Algebra the commutator Table for V_i is a $(4 \otimes 4)$ table whose (i, j)th entry express the Lie Bracket $[V_i, V_j]$ given by the above Lie Algebra L. The Table 1 is skew-symmetric and the diagonal elements all vanish. The coefficient $C_{i,j,k}$ is the coefficient of V_i of the (i, j)th entry of the Commutator Table and the related structure constants can be easily calculated from the above table as follows

$$C_{2,4,1} = -1, \text{ and } C_{4,2,1} = 1$$

The Lie Algebra L is Solvable. The Radical of g is $< V_3 > \oplus < V_1, V_2, V_4 >$.

In the next section, we derive the reduction of (7) to PDEs with two independent variables and ODEs. These are four one-dimensional Lie subalgebras

$$L_{s,1} = \{V_1\}\ ,\ L_{s,2} = \{V_2\}\ ,\ L_{s,3} = \{V_3\}\ ,\ L_{s,4} = \{V_4\}$$

and corresponding to each one-dimensional subalgebras we may reduce (7) to a PDE with two independent variables.

Further reductions to ODEs are associated with two-dimensional subalgebras. It is evident from the commutator table that there is no two-dimensional solvable non-abelian subalgebras. And there are five two-dimensional Abelian subalgebras, namely, $L_{A,1} = \{V_1, V_2\}$, $L_{A,2} = \{V_1, V_3\}$, $L_{A,3} = \{V_1, V_4\}$, $L_{A,4} = \{V_2, V_3\}$, and $L_{A,5} = \{V_3, V_4\}$.

2.1. Reductions of $u_t + uu_x + u_x^2 - Au_{xx} - Bu_{yy} = 0$ by One-dimensional Subalgebra

Case A: The Subalgebra $V_1 = \partial_x$. The characteristic equation associated with this generator is

$$\frac{dx}{1} = \frac{dy}{0} = \frac{dt}{0} = \frac{du}{0}\ .$$

Thus we integrate the characteristic equation to get these similarity variables,

$$y = s, \quad t = r \text{ and } u = w(r, s). \tag{17}$$

Using these similarity variables in Eq.(7) can be transformed in the form

$$w_r - Bw_{ss} = 0. \tag{18}$$

Where B is a constant.

Case B: The Subalgebra $V_2 = t\partial_x + \partial_u$. The characteristic equation associated with this generator is

$$\frac{dx}{t} = \frac{dy}{0} = \frac{dt}{0} = \frac{du}{1}\ .$$

Thus we integrate the characteristic equation to get these similarity variables,

$$y = s, \quad t = r \text{ and } w = tu - x\ . \tag{19}$$

Using these similarity variables in Eq.(7) can be transformed in the form

$$rw_r + 1 - r^2 Bw_{ss} = 0. \tag{20}$$

where B is a constant.

Case C: The Subalgebra $V_3 = \partial_y$. The characteristic equation associated with this generator is

$$\frac{dx}{0} = \frac{dy}{1} = \frac{dt}{0} = \frac{du}{0}\ .$$

Thus we integrate the characteristic equation to get these similarity variables,

$$x = s, \quad t = r \text{ and } u = w(r, s)\ . \tag{21}$$

Using these similarity variables in Eq.(7) can be recast in the form

$$w_r + ww_s + w_s^2 - Aw_{ss} = 0. \tag{22}$$

where A is a constant.

Case D: The Subalgebra $V_4 = \partial_t$. The characteristic equation associated with this generator is

$$\frac{dx}{0} = \frac{dy}{0} = \frac{dt}{1} = \frac{du}{0}\ .$$

Thus we integrate the characteristic equation to get these similarity variables,

$$s = x, \quad y = r \text{ and } u = w(r, s)\ . \tag{23}$$

Using these similarity variables in Eq.(7) can be transformed in the form

$$ww_s + w_s^2 - Aw_{ss} - Bw_{rr} = 0. \tag{24}$$

where A and B are constants.

2.2. Reductions of $u_t + uu_x + u_x^2 - Au_{xx} - Bu_{yy} = 0$ by two-dimensional abelian subalgebra

Case 1: The Subalgebra V_1 and V_3.

In this case, we find that the given generators commute $[V_1, V_3] = 0$. Thus either of V_1 or V_3 can be used to start the reduction with. For our purpose we begin reduction with V_1. The transformed equation is given in Eq 18 At this stage, we express V_3 in terms of the similarity variables defined in (16). It is straight-forward to note that V_3 in the new variables takes the form

$$\tilde{V_3} = \partial s. \tag{25}$$

The characteristic equation for $\tilde{V_3}$ is

$$\frac{ds}{1} = \frac{dr}{0} = \frac{dw}{0}.$$

Integrating this equation as before leads to new variables $\alpha = r,\ \beta(\alpha) = w$, which reduces (18) to a second order differential equations

$$\beta^{'} = 0. \tag{26}$$

Case 2: The Subalgebra V_3 and V_4

In this case, we find that the given generators commute $[V_3, V_4] = 0$. Thus either of V_3 or V_4 can be used to start the reduction with. For our purpose we begin reduction with V_3. The transformed equation is given in Eq(22). At this stage, we express V_4 in terms of the similarity variables defined in (16). It is straight-forward to note that V_4 in the new variables takes the form

$$\tilde{V_4} = \partial r. \tag{27}$$

The characteristic equation for $\tilde{V_4}$ is

$$\frac{ds}{0} = \frac{dr}{1} = \frac{dw}{0}.$$

Integrating this equation as before leads to new variables $\alpha = s,\ \beta(\alpha) = w$, which reduces (22) to a second order differential equations

$$\beta\beta^{'} + \beta^{'2} - A\beta^{''} = 0. \tag{28}$$

Remaining reductions are given in the form of Appendix I.

3. Symmetry Group of $u_t + uu_x + u_x^2 - Au_{xx} - e^{\alpha t}\sigma u_{yy} = 0$

Following the procedure adopted in above cases. It is easy to find that the components $\xi,\ \eta,\ \tau,\ \phi$ of infinitesimal generators of V becomes,

$$\begin{aligned} \xi &= c_4 t + c_3, \\ \eta &= c_1 + c_2\frac{y\alpha}{2}, \\ \tau &= c_2, \\ \text{and } \phi &= c_4, \end{aligned} \tag{29}$$

where $c_1,\ c_2,\ c_3$ and c_4 are constants. Thus, the Lie algebra of infinitesimal symmetries of the Potential Burgers Equation is spanned by the four vector fields

$$\begin{aligned} V_1 &= \partial_y, \\ V_2 &= \frac{y\alpha}{2}\partial_y + \partial_t, \\ V_3 &= \partial_x, \\ V_4 &= t\partial_x + \partial_u. \end{aligned} \tag{30}$$

It is easy to check that the symmetry generators found in (29) form a closed Lie algebra whose commutation relations are given in Table 2. The one-parameter groups $g_i(\epsilon)$ generated by the V_i where $i = 1, 2, 3, 4$ are

$$\begin{aligned} g_1(\epsilon) : (x, y, t; u) &\rightarrow (x, y + \epsilon, t, u)\ , \\ g_2(\epsilon) : (x, y, t; u) &\rightarrow (x, y + \frac{y\alpha}{2}\epsilon, t + \epsilon, u)\ , \\ g_3(\epsilon) : (x, y, t; u) &\rightarrow (x + \epsilon, y, t, u)\ , \\ g_4(\epsilon) : (x, y, t; u) &\rightarrow (x + t\epsilon, y, t, u + \epsilon) \end{aligned}$$

where $exp(\epsilon V_i)\ (x, y, t; u) = (\bar{x}, \bar{y}, \bar{t}; \bar{u})$ and

(i) $g_1,\ g_3$ and g_4 are the space-invariant,

(ii) g_2 is time translation of the equation.

Table 2. **Commutator Table**

$[V_i, V_j]$	V_1	V_2	V_3	V_4
V_1	0	$-\frac{\alpha}{2}V_1$	0	0
V_2	$\frac{\alpha}{2}V_1$	0	0	$-V_3$
V_3	0	0	0	0
V_4	0	V_3	0	0

The commutation relations of the Lie algebra L, determined by $V_1,\ V_2,\ V_3$ and V_4 are shown in the above Table

$$[V_2, V_4] = -V_3 \text{ and } [V_4, V_2] = V_3$$

For this two-dimensional Lie Algebra the commutator Table for V_i is a $(4 \otimes 4)$ table whose (i, j)th entry express the Lie Bracket $[V_i, V_j]$ given by the above Lie Algebra L. The Table is skew-symmetric and the diagonal elements all vanish. The coefficient $C_{i,j,k}$ is the coefficient of V_i of the (i, j)th entry of the Commutator Table and the related structure constants can be easily calculated from the above table as follows

$$C_{2,4,3} = -1, \text{ and } C_{4,2,3} = 1$$

The Lie Algebra L is Solvable. The Radical of g is $< V_1, V_3 > \oplus < V_2, V_4 >$.

In the next section, we derive the reduction of (8) to PDEs with two independent variables and ODEs. These are four one-dimensional Subalgebras

$$L_{s,1} = \{V_1\} \ , \ L_{s,2} = \{V_2\} \ , \ L_{s,3} = \{V_3\} \ , \ L_{s,4} = \{V_4\}$$

and corresponding to each one-dimensional subalgebras we may reduce (8) to a PDE with two independent variables. Further reductions to ODEs are associated with two-dimensional subalgebras. It is evident from the commutator table that there is only one two-dimensional solvable non-abelian subalgebras. And there are three two-dimensional Abelian Subalgebras, namely, $L_{A,1} = \{V_1, V_3\}$, $L_{A,2} = \{V_1, V_4\}$, $L_{A,3} = \{V_2, V_3\}$, $L_{A,4} = \{V_3, V_4\}$, and $L_{NA,1} = \{V_1, V_2\}$

3.1. Reductions of $u_t + uu_x + u_x^2 - Au_{xx} - e^{\alpha t}\sigma u_{yy} = 0$ by One-dimensional Subalgebra

Case A: The Subalgebra $V_1 = \partial_y$. The characteristic equation associated with this generator is

$$\frac{dx}{0} = \frac{dy}{1} = \frac{dt}{0} = \frac{du}{0} \ .$$

Thus we integrate the characteristic equation to get these similarity variables,

$$x = s, \quad t = r \text{ and } u = w(r, s). \tag{31}$$

Using these similarity variables in Eq.(8) can be transformed in the form

$$w_r + ww_s + w_s^2 - Aw_{ss} = 0. \tag{32}$$

where A is a constant.

Case B: The Subalgebra $V_2 = \frac{y\alpha}{2}\partial_y + \partial_t$. The characteristic equation associated with this generator is

$$\frac{dx}{0} = 2\frac{dy}{y\alpha} = \frac{dt}{1} = \frac{du}{0} \ .$$

Thus we integrate the characteristic equation to get these similarity variables,

$$x = s, \quad r = y^{-2}e^{\alpha t} \text{ and } w = u \ . \tag{33}$$

Using these similarity variables in Eq.(8) can be transformed in the form

$$ww_s + w_s^2 - Aw_{ss} = 0. \tag{34}$$

where A is a constant.

Case C: The Subalgebra $V_3 = \partial_x$. The characteristic equation associated with this generator is

$$\frac{dx}{1} = \frac{dy}{0} = \frac{dt}{0} = \frac{du}{0} \ .$$

Thus we integrate the characteristic equation to get these similarity variables,

$$y = s, \quad t = r \text{ and } u = w(r, s) \ . \tag{35}$$

Using these similarity variables in Eq.(8) can be recast in the form

$$w_r - e^{\alpha r}\sigma w_{ss} = 0. \tag{36}$$

Case D: The Subalgebra $V_4 = t\partial_x + \partial_u$. The characteristic equation associated with this generator is

$$\frac{dx}{t} = \frac{dy}{0} = \frac{dt}{0} = \frac{du}{1} \ .$$

Thus we integrate the characteristic equation to get three similarity variables,

$$s = y, \quad t = r \text{ and } w = tu - x \ . \tag{37}$$

Using these similarity variables in Eq.(8) can be recast in the form

$$rw_r + 1 - r^2 e^{\alpha r}\sigma w_{ss} = 0. \tag{38}$$

3.2. Reductions of $u_t + uu_x + u_x^2 - Au_{xx} - e^{\alpha t}\sigma u_{yy} = 0$ by Two-dimensional Abelian Subalgebra

Case 1: The Subalgebra V_1 *and* V_3

In this case, we find that the given generators commute $[V_1, V_3] = 0$. Thus either of V_1 or V_3 can be used to start the reduction with. For our purpose we begin reduction with V_1. The transformed equation is given in Eq (32). At this stage, we express V_3 in terms of the similarity variables defined in (30). It is straight-forward to note that V_3 in the new variables takes the form

$$\tilde{V_3} = \partial s. \tag{39}$$

The characteristic equation for $\tilde{V_3}$ is

$$\frac{ds}{1} = \frac{dr}{0} = \frac{dw}{0}. \tag{40}$$

Integrating this equation as before leads to new variables $\alpha = r$, $\beta(\alpha) = w$, which reduces (32) to a first order differential equations

$$\beta^{'} = 0. \tag{41}$$

3.3. Reductions of $u_t + uu_x + u_x^2 - Au_{xx} - e^{\alpha t}\sigma u_{yy} = 0$ by two-dimensional Non-abelian Subalgebra

Case 2: The Subalgebra V_1 and V_2

From Table 2, we find that the two symmetry generators V_1 and V_2 satisfy the commutation relation $[V_1, V_2] = \frac{\alpha}{2}V_1$. This suggests reduction in this case should start with V_1. The transformed equation is given in Eq (32). The transformed V_2 is

$$\tilde{V_2} = \partial_r. \tag{42}$$

The invariants of $\tilde{V_2}$ are $\alpha = s$ and $\beta = w$ which reduces (32) to the ODE,

$$\beta\beta^{'} + \beta^{'2} = 0. \tag{43}$$

Remaining reductions are given in the form of Appendix II.

Appendix I

Algebra	Reduction
$[V_1, V_3] = 0$	$\beta' = 0$
$[V_1, V_4] = 0$	$-B\beta' = 0$
$[V_2, V_3] = 0$	$\alpha\beta' + 1 = 0$
$[V_3, V_4] = 0$	$\beta\beta' + \beta'^2 - A\beta'' = 0$

Appendix II

Algebra	Reduction
$[V_1, V_3] = 0$	$\beta' = 0$
$[V_1, V_4] = 0$	$\alpha\beta' + 1 = 0$
$[V_1, V_2] = 0$	$\beta\beta' + \beta'^2 - A\beta'' = 0$

4. Conclusion

In this Paper,

(i) A $(2+1)$-dimensional variable cofficient potential Burgers equation $u_t + uu_x + u_x^2 - A(t)u_{xx} - B(t)u_{yy} = 0$ where $A(t)$ and $B(t)$ are classified through group classification technique.

(ii) Equations (7) and (8) admits a two-dimensional symmetry group.

(iii) It is established that the symmetry generators form a closed Lie algebra.

(iv) Classifications of Symmetry algebras of (7) and (8) into one- and two-dimensional abelian subalgebras are carried out.

(v) Systematic reductions to (1+1)-dimensional PDE and then to first- or second order ODEs are performed using one-dimensional and two-dimensional solvable Abelian subalgebras.

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